Finite difference analysis for combined pile raft foundations under vertical loads

Der-Wen Chang[1], H.-W. Lien[1], and T.-Y. Wang[2]

[1] Department of Civil Engineering, Tamkang University, 151, Yin-Chuan Rd., Tamsui, New Taipei City, 25137, Taiwan

ABSTRACT

A newly proposed three-dimensional analysis is introduced herein to estimate the foundation settlements of a vertically loaded piled raft foundation on the ground surface. Thin-plate theory with the effects of boundaries was adopted whereas alternative soil springs underneath the raft were studied for the optimal modeling. The pile-soil resistances were computed solving the equivalent stiffness of piles and surrounding soils based on the discrete wave equations. The analysis was examined with the three-dimensional finite element analysis. Although the new analysis can provide rational predictions, some significant errors were found for foundation settlements at the corner with the effects of the pile-to-pile interactions. Both discrepancies were aimed to be solved in the proposed analysis.

Keywords: finite difference analysis; combined pile raft foundation; vertical load

1 INTRODUCTION

The settlements of combined pile raft foundation (CPRF) can be modeled by two-dimensional (2D) or three-dimensional (3D) analyses. For 2D analysis, the foundation can be treated as a one-dimensional (1D) beam on soil and pile-soil springs (see Fig. 1). The analysis is termed as Beam on Elastic Foundation or Winkler foundation. This type of solution is applicable when the length-to-width ratio (L/W) of the raft (where L is the length, W is the width) exceeds 10. The 2D analysis has been discussed for decades (Biot 1937; Mathews 1958; Bowles 1977; Ting and Mockry 1984; Jones 1997; Chen 1998; Tomlinson and Boorman 2001; Dinev 2012; Chiou *et al.* 2016; Chang *et al.* 2016). The applications of versatile springs of the soils and the pile-soil elements became the keys to the adequateness of the solutions.

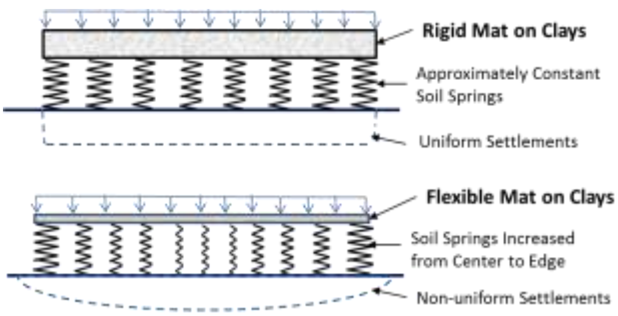

Fig. 1D beam model for raft foundation analysis

The 3D analysis is modeled taking the raft as a two-dimensional plate (or mat) on the ground. Analytical formulations have been presented (Timoshenko and Krieger 1959; Vlasov and Leontev 1966; Kukreti and Ko 1992). Owing to complexities of the solutions, they are rarely used in engineering practice. Numerical solution based on a series of connecting strip footings has been suggested by Poulos (1991), alternate solution was brought by Poulos (1994) with the solutions of plate on soil continuums from boundary integrals. In general, 3D analysis can be found in many studies (Randolph 1983; Clancy and Randolph 1993; Horikoshi and Randolph 1996; Kobayashi *et al.* 2009; Kitiyodom and Matsumoto 2002; Kitiyodom *et al.* 2005). Fig. 2 shows the typical model of the 3D simulations from Clancy and Randolph (1993).

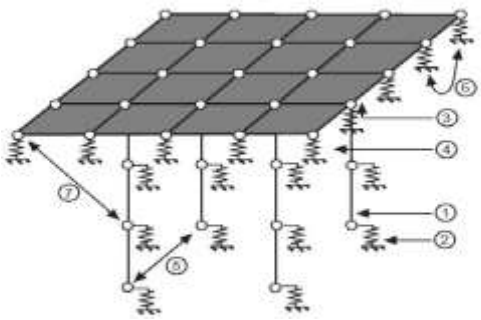

Fig. 2 Piled raft foundation with 2D raft (from Clancy and Randolph, 1993)

The settlements of CPRF can be solved using either equations of motion formed by characteristic matrices of the structural elements or wave equations derived for the structural system. The former solves the foundation settlements based on stiffness matrix of the foundation (and the mass/damping matrices for dynamic problems) while the later often use the difference formula to solve the foundation settlements. 2D raft can be modeled by either finite elements or simplified grid (beam-column) elements underlain by a set of applicable soil springs and pile-soil elements.

Correspondent discrete solution was suggested by Bowles (1977) with the finite difference scheme on the wave equations. Such analysis is applicable to an infinite raft or a rigid raft where the foundation settlements are nearly uniform. For flexible foundation where the differential settlements became significant, the solution suggested by Bowles (1977) needs modifications. The soils and pile-soil elements can be attached to the raft in order to simulate the resistances of the foundation.

With such concern, this study proposed a 3D FDA for a surface raft foundation subjected to vertically uniform static load. The governing equations adopted from the thin plate theory were modified with boundary values where the moments and shears were vanished. The equivalent stiffness of the pile-soil elements were computed and adopted together with the soil springs to support the raft foundation. The proposed analyses were verified with three-dimensional FEM analysis to ensure its application.

2 METHODOLOGIES

Theory of Plate can be categorized for thin plate and thick plate. In general if the thickness of the plate (D) is less than a tenth of the width (W) of plate, it can be treated as thin-plate. The Kirchhoff-Love classical plate theory was suggested on thin plate. The thick plate theory considers the in-plane shear strains whereas the thin plate theory does not.

2.1 Governing Equation

According to Timoshenko and Woinowsky-Krieger (1959), governing equation of the vertical displacements of a thin plate subjected to vertically uniform load (q) and point load (P) can be written as follows,

$$\frac{\partial^4 w}{\partial x^4} + \frac{2\,\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{12q\left(1-\nu^2\right)}{ED^3} + \frac{12P\left(1-\nu^2\right)}{ED^3(\partial x\,\partial y)} \tag{1}$$

where w is the vertical displacement of the raft, ν and E are the Poisson's ratio and Young's Modulus of raft, D is thickness of the raft, and x and y are the spatial variables.

2.2 Boundary Conditions

For a raft foundation located at the ground surface as shown in Fig. 3, the moments and shear forces are assumed vanished at edge of the foundation. The top and bottom edges of the raft where y=constant, Mx (bending moment rotating at the x-direction) and Vy (vertical shear force at the surface normal to the y-direction) can be written as follows,

$$M_x = -B(\partial^2 w/\partial y^2 + \nu\,\partial^2 w/\partial x^2) = 0 \tag{2}$$

$$V_y = -B\left[\frac{\partial^3 w}{\partial y^3} + \frac{(2-\nu)\,\partial^3 w}{\partial y\,\partial x^2}\right] = 0 \tag{3}$$

where B is the expression of $ED^3/(12(1-\nu^2))$. Similarly, at the left and right edges of the raft where x=constant, the boundary conditions M_y and V_x are:

$$M_y = -B(\partial^2 w/\partial x^2 + \nu\,\partial^2 w/\partial y^2) = 0 \tag{4}$$

$$V_x = -B\left[\frac{\partial^3 w}{\partial x^3} + \frac{(2-\nu)\,\partial^3 w}{\partial x\,\partial y^2}\right] = 0 \tag{5}$$

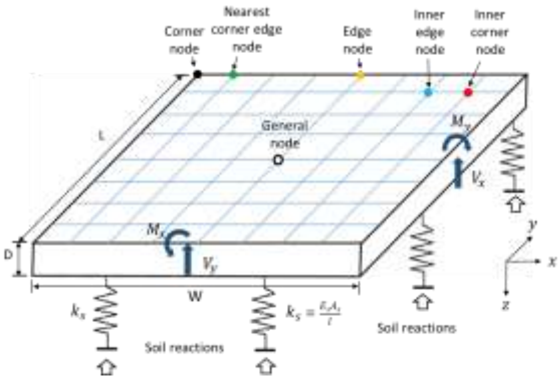

Fig. 3 Layout and discrete nodes of a vertically loaded raft foundation

2.3 Soil Springs

For the soil resistance underneath the raft, a number of models can be used. For simplicity, the single rod stiffness and Lysmer's analog model were used in this study. If rod stiffness model was used, soil spring constant K_s is calculated as E_sA_s/l where E_s is the Young's Modulus of the soil, A_s is the effective area of the soils underneath the raft, and l is the length of the soil spring (or effective thickness of the soils). By taking the underneath soil reaction effects similar to the pressures of the superstructure loads, Eq. (1) can be rewritten by replacing q with q^* where $q^* = q - \Sigma K_s w_k / A_r = q-(E_s/l)\Sigma A_{sk\times} w_k / A_r$; w_k is foundation settlement at the k^{th} node, A_{sk} is the area of soil spring under the k^{th} node, and A_r is the total area of the raft which equals to ΣA_{rk}, where A_{rk} stands for the area of raft at the k^{th} node. Defining q_k^* as the modified load allocated at the k^{th} node, for simplicity, q_k^* can be approximated as $q-(E_s/l)w_k(A_{sk}/A_{rk})$, where (A_{sk}/A_{rk}) is called as the area ratio at the k^{th} node.

The above simulations would match closely to the flexible foundations where the raft size is relatively large and the soil springs are varied underneath the raft. For smaller raft foundation that behaves more rigidly, this assumption should be not applicable. One can adopt other types of soil spring model. For example, the Lysmer's Analog soil spring can be used. The total soil spring constant of the whole raft foundation could be calculated as $4G_sr_o/(1-\nu_s)$, where G_s is the shear modulus of the soil, ν_s is the Poisson's ratio of the soil, and r_o is the equivalent radius of the raft foundation.

2.4 Pile-Soil Elements

For piles underneath the raft, the equivalent stiffness of the piles was computed assuming linearly elastic soil springs attached to the piles (See Fig. 4). The single piles were analyzed assuming unit load acting on top of the pile. With the discrete FD formulas for the wave equations on the pile segments (Chang and Lin 1999), the pile displacements can be solved.

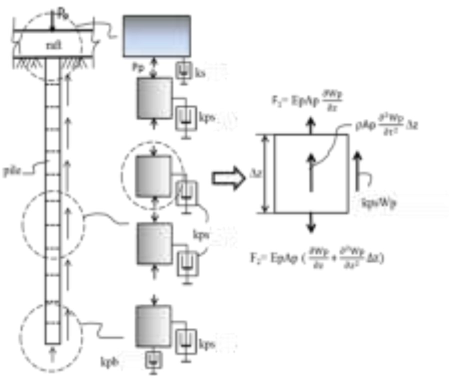

Fig. 4 Pile-soil elements from discrete wave equation

For linear elastic pile behaviors, the equivalent stiffness, kp of the pile can be easily obtained dividing the load by the displacement. Simplified soil springs (*e.g.*, G_sA_s/l_s and E_sA_s/l_s) were assumed in the study for soils at the pile shaft and the pile tip. The equivalent pile stiffness was examined with various soil models. They were found very similar with each other. See Table 1. The pile-soil elements can be combined with the soil springs to model the reactions under the raft. Therefore, Eq. (1) for the nodes where the pile locates can be rewritten as follows,

$$\frac{\partial^4 w}{\partial x^4}+\frac{2\partial^4 w}{\partial x^2 \partial y^2}+\frac{\partial^4 w}{\partial y^4}=\frac{12q''(1-v^2)}{Eh^3}-\frac{\rho}{E}\frac{\partial^2 W}{\partial t^2} \tag{6}$$

where q" is expressed as $q'' = q- (Ks+Kp)\times w_k/A_r$. Thus, the CPRF settlements can be calculated using the modified load intensity, q".

Table 1 Comparison of the equivalent stiffness calculated from various elastic soil models

Model	K_s along shaft	K_b at bottom	Kp (kN/m)
Rod stiffness	$K_s=\frac{G_sA}{l_s}$ (l_s=1m)	$K_b=\frac{E_bA}{l_b}$ (l_b=1m)	130680
Liang (1993)	$K_s=\frac{2\pi L_i G_s}{\ln(2.5L_i(1-\mu_s)/r_0)}$	$K_b=\frac{\pi r_0 E}{2(1-\mu_s^2)I_b}$	144840
Matsumoto (2013)	$K_p=\frac{2\pi G\Delta L}{\ln(r_m/r_0)}$		131208

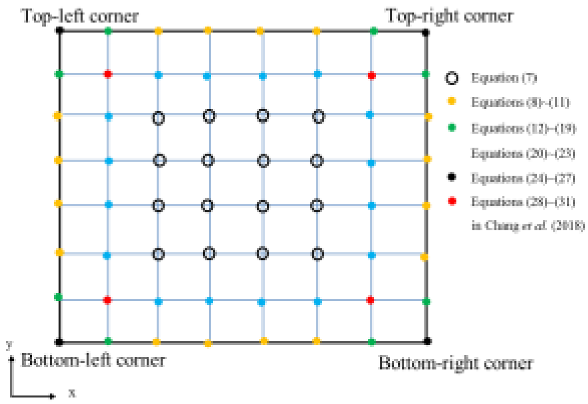

Fig. 5 Allocations of the formulas used at the nodes of raft

3 FORMULAS AND PROGRAMS

Using the central-difference formulas, the resulting formulations for the nodes at a surface foundation can be derived. The orientation and categories of these nodal points are shown in Fig. 5. Details of the derivations can be found in Chang et al. (2018). Note that the spacing distance between the nodes in *x*- and *y*-directions are kept the same (i.e., $\Delta x=\Delta y=s$) for simplicity of the expressions. Moreover, the point load P applied at arbitrary nodes of the raft can be taken as an extra uniform load applied to that node within the area which is equal to $\Delta x \times \Delta y$. Fig. 6 shows the nodal points used and the fictitious points encountered in the derivations. With the discrete equations derived, one can easily establish a set of dependent equations for the nodes allocated at the raft. Matrix analysis is required to solve for the foundation displacements.

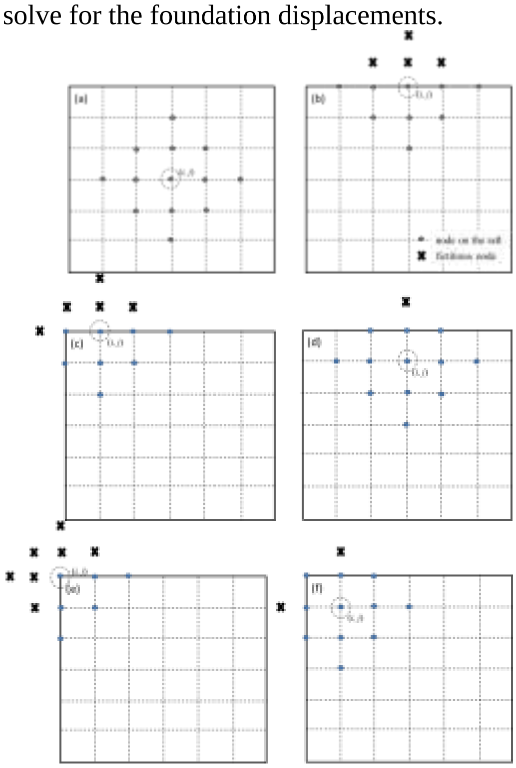

Fig. 6 Nodes with the fictitious points to be eliminated

When calculating the soil reactions at the nodes along the edges, the area ratio (A_{sk}/A_{rk}) of the soil springs can be represented by a value, n defined by dividing the length of the area with the standard width (which is equal to 1.0 m) of the soils underneath the general nodal points. For the nodes at the corners, the area ratio of the soil spring would be expanded to $n\times n/1.0$ as n^2. The spacing distance between two adjacent nodes is kept as 1 m for simplicity. Fig. 7 depicts the area of soil springs for nodes at the edges and the corners. As a result, a computer program WERAFT-S was suggested for the raft foundation. With the use of the pile-soil elements, a more advanced computer program WEAPR-S was developed for CPRF (Lien 2018).

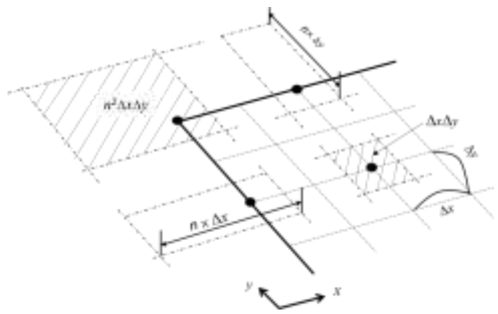

Fig. 7 Effective area of the soils at nodes along the edges

4. VALIDATION AND COMPARISONS

The results from WERAFT-S analysis with the rod stiffness of the soils on an artificial CPRF are shown in Fig. 8 and Table 2. Table 3 lists the material parameters and the dimensions of the model in the analysis. The comparisons were made using the Midas-GTS NX program (Midas 2017). It was found that the optimal thickness (l) for rod stiffness of the soil is approximately around 20-22 meters. Area ratio (n) of 2.5 for the nodes at the edge will provide comparable results if the rod stiffness model was used. If the Lysmer's analog model was adopted where the total foundation stiffness is equally distributed to the nodes, the foundation settlements calculated at the corners would be smaller compared to the ones from Midas-GTS NX analysis, and the results are depending on the soils. Detailed discussions on the parametric studies can be found in Chang et al (2018).

Table 2 Comparisons of raft settlements at various locations

Fdt. Settlement	Midas-GTS NX	WERAFT-S	
		Rod Spring	Lysmer Spring
Center	1.88 cm	1.86 cm	1.86 cm
Edge	1.31 cm	1.37 cm	1.36 cm
Corner	0.97 cm	0.92 cm	0.79 cm

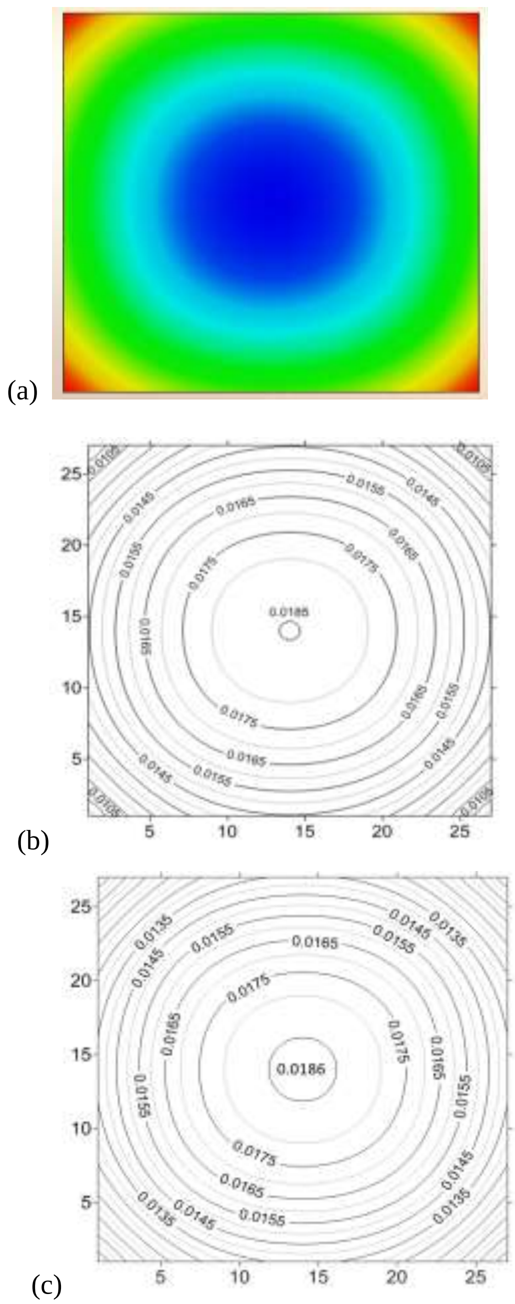

Fig. 8 Contour plots for foundation settlements from WERAFT-S and Midas-GTS NX analysis (a) Midas (b) WERAFT-S with rod springs (c) WERAFT-S with Lysmer springs

Table 3 Numerical model parameters and dimensions

Soils	Shear wave velocity = 150m/s, ν=0.4, γ_s = 19 kN/m^3
Foundation	Concrete raft : 26m×26m×1m Concrete piles : round pile w/ diameter 1m and length 30m E_c=3x10^{4}Mpa, γ_c=24kN/m^3, ν=0.15
Load	Uniform load with intensity of 100 kPa

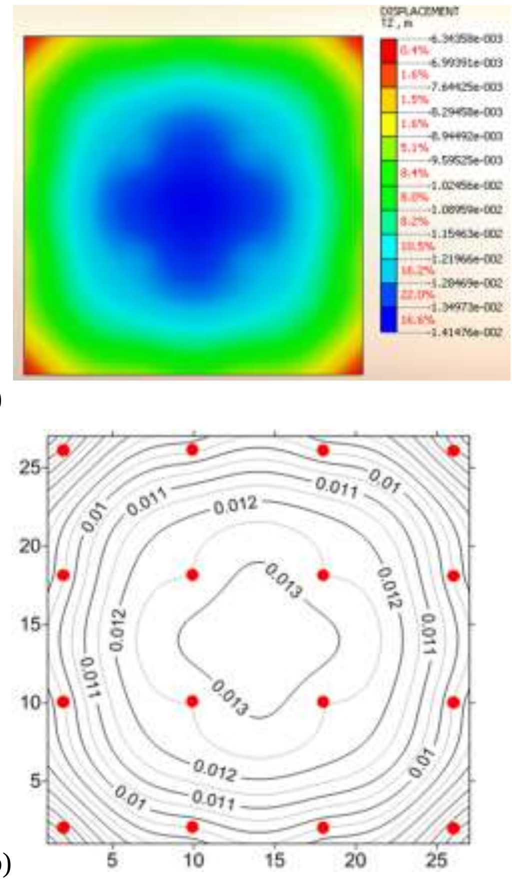

Fig. 9 Contour plots for foundation settlements from WEAPR-S and Midas-GTS NX analysis (a) Midas (b) WEAPR-S with Lysmer springs

Table 4 Comparisons of CPRF settlements at various locations

Fdt. Settlement	Midas-GTS NX	WEAPR-S
Center	1.41 cm	1.35 cm
Edge	0.96 cm	0.95 cm
Corner	0.64 cm	0.34 cm

In comparing the solutions from WEAPR-S (using Lysmer's analog model) with those from Midas analysis, the calculated foundation settlements at the center and the edges were found similar when S/d is equal to 8. Again, the ones at the corners from WEAPR-S were found nearly half of those calculated from the Midas analysis. (See Fig. 9 and Table 4) The differences appearing at the settlements of the corner were also found when the ground stiffness and was changed (see Fig. 10).

In addition, it was learnt that the pile-to-pile interactions are significant when S/d is less than 8 (see Fig. 11). Without the simulations of pile-to-pile interactions, the foundation settlements estimated by WEAPR-S would be much smaller than those obtained from the Midas analysis.

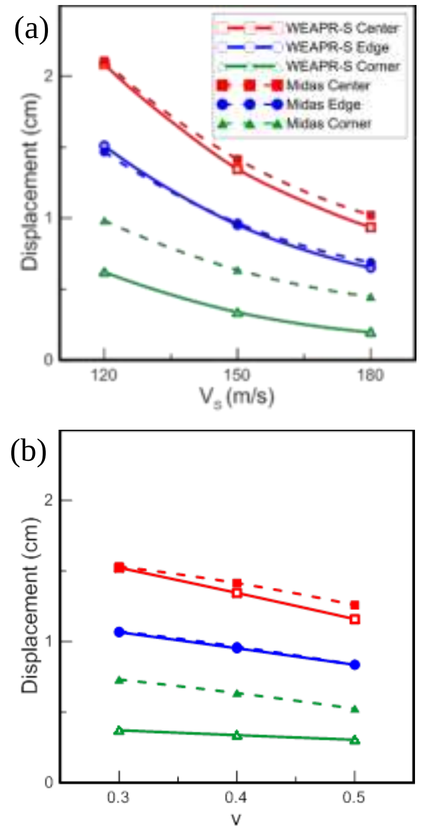

Fig.10 Effects of soil stiffness on foundation settlements (a) shear wave velocity in the range of 120-180 m/s (b) Poisson's ratio of 0.3-0.5

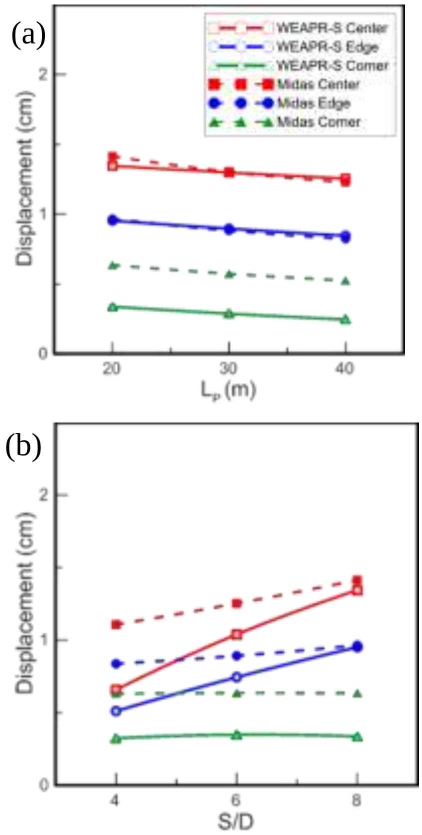

Fig.11 Effects of pile length and pile orientations on foundation settlements (a) length in between 20m-40m (b) S/D in the range of 4-8

5 CONCLUDING REMARKS

This paper presents the three dimensional modeling of the settlements of a combined pile raft foundation (CPRF) under vertically uniform loads. Finite width of a square foundation was monitored with proper boundary influences. Finite difference formulas were used to model the foundation settlements. The analysis was found to provide rational results in comparison with the 3D FEM analysis for settlements at the center and the edge of foundation. The settlements obtained at the corner were found much less than those suggested by the FEM analysis. The reason behind this is possibly due to the finite thickness of the soil layer used in the FEM analysis whereas the proposed analysis assumes the foundation was resting on the surface of an elastic half-space. Another important factor is that the soil stiffness should vary underneath the raft as a flexible foundation. The deviation may be caused by treating the soil stiffness equally underneath the foundation (which corresponds to the Lysmer's analog springs). The drawback of such modeling can be improved by using the rod stiffness for the soils with the enlarged areas at the edge. The estimations were found more agreeable even at the corner of the foundation. As to the settlements of combined pile raft foundation, it was found that the pile-to-pile interactions will become significant by reducing the pile-to-pile spacing distance (*i.e.*,S/D<8). Such mechanism must be taken into account in the proposed analysis. The above findings can be referred to improve the proposed analyses.

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