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Stress-strain theories for normally consolidated clays

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The stress-strain theory of Roscoe and Poorooshasb was presented in a modified form to enable it to be compared with the other theories (Cam-clay theory, Modified theory, and the Revised theory) developed at Cambridge based on energy balance equations and the concepts of plasticity. Conditions were stated and verified experimentally to illustrate when the Cam-clay theory or the Modified theory or the Revised theory will predict identical strains to those of Roscoe and Poorooshasb's theory. All the theories are expressed in mathematically similar forms for ease of comparison.

SYMBOLS

e_o, e	initial and current voids ratio
F, f_1, f_2	functions defined with respect to Roscoe and Poorooshasb theory
F^*, f_1^*, f_2^*	functions defined with respect to Roscoe, Schofield and Thurairajah theory
F^{**}, f_1^{**}, f_2^{**}	functions defined with respect to Roscoe and Burland theory
k	slope of isotropic swelling line in (e, log p) plot
L_o, L	initial and current height of sample
M	slope of critical state line in (q, p) plot
p	mean normal stress
P_o	isotropic consolidation stress
P_e	mean equivalent pressure $= p_o \exp \dfrac{(e_o - e)}{\lambda}$
q	deviator stress
v	volumetric strain
V_o, V	initial and current volume of sample
σ_1', σ_2', σ_3'	principal effective compressive stresses
ϵ_1, ϵ_2, ϵ_3	principal compressive strains
ϵ	shear strain
$\bar{\epsilon}$	shear strain associated with constant q yield loci
λ	slope of isotropic consolidation line in(e, log p) plot
η	stress ratio $= q/p$
ϕ	function associated with the shear strain from constant q yield loci
$\phi'(\eta)$	$\dfrac{d}{d\eta}\phi(\eta)$

INTRODUCTION

This paper is concerned with the comparison of the observed deformation behaviour (of over 30 normally consolidated specimens of kaolin sheared along a wide variety of stress paths) with those predicted from a number of stress-strain theories developed at Cambridge. The investigation of the stress-strain behaviour is restricted to the simple axi-symmetric conditions of the conventional triaxial system of stresses on cylindrical specimens wherein the principal axes of stress and strain are at all times compelled to coincide and the intermediate stress is always equal to the major or minor principal stress. The stress-strain theories considered are (i) the incremental stress-strain theory of Roscoe and Poorooshasb (1963), (ii) the stress-strain theory of Roscoe, Schofield and Thurairajah (1963) based on an energy balance and the concepts of plasticity and (iii) the Revised theory of Roscoe and Burland (1968) based on the concept of singular yield surfaces (see Koiter (1953) and Saunders (1955)). The incremental stress-strain theory of Roscoe and Poorooshasb (1963) is presented in a slightly different form to facilitate comparison with the other two theories. Conditions were stated and verified experimentally to illustrate when the Cam-clay theory or the Revised theory would predict identical strains to those of Roscoe and Poorooshasb theory. The investigation revealed an important finding that while all the four theories could be expressed mathematically in a similar form it is only the energy equation used in the Revised theory that can give an expression which is able to predict identical volumetric strains to those of Roscoe and Poorooshasb (1963). Also the additional constant q (where q is the deviator stress) yield locus, used in the Revised theory is necessary for the successful prediction of shear strains.

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STRESS AND STRAIN PARAMETERS

The stress parameters used in the analysis of triaxial test results are

$$p = \frac{(\sigma_1' + 2\sigma_3')}{3}$$

$$q = (\sigma_1' - \sigma_3') \text{ since } \sigma_2' = \sigma_3'$$

where σ_1', σ_2', and σ_3' are the principal compressive effective stresses. In terms of the principal compressive strains ϵ_1, ϵ_2 and ϵ_3 the relevant strain parameters for use under the axi-symmetric conditions of the triaxial test are

$$v = (\epsilon_1 + 2\epsilon_3)$$

$$\text{and } \epsilon = \frac{2(\epsilon_1 - \epsilon_3)}{3} \quad \text{since } \epsilon_2 = \epsilon_3$$

The axial strain, ϵ_1 is defined as

$$\epsilon_1 = \int_L^{L_o} \frac{dL}{L} = \log\left(\frac{L_o}{L}\right) \text{ (compression being + ve)}$$

L_o is the initial height and L, the current height and the volumetric strain v is

$$v = \int_V^{V_o} \frac{dV}{V} = \log\left(\frac{V_o}{V}\right) \text{ (compression being + ve)}$$

V_o is the initial volume and V the current volume or expressed in terms of the voids ratio (e) is

$$v = \log\left(\frac{1 + e_o}{1 + e}\right)$$

where e_o is the voids ratio of the sample at the end of isotropic consolidation. In the present paper unless otherwise stated v and ϵ will be measured from the state of the sample at the end of its preparation under isotropic stress conditions and just prior to shear.

The state boundary surface in p, q, e space (Roscoe and Poorooshasb, 1963) is that surface confining a space between itself and the origin within which a point can represent a state of an element of the soil but outside which a point cannot represent such a state. Any point of the soil specimen in the p, q, e space is called a state point and the path followed during a test is called the state path.

MATERIAL TESTED SAMPLE PREPARATION AND TESTING PROCEDURES

All specimens were prepared from air-dried kaolin (liquid limit 74% plastic limit 42% and specific gravity 2.6) mixed with water to a slurry of 160% moisture content. The slurry was one dimensionally consolidated in a special former to a maximum pressure of 156.0 kNm^{-2}. Subsequently the former was removed and the sample was isotropically consolidated to the required cell pressure. For a detailed description of the sample preparation and testing procedure see Balasubramaniam (1969). The effect of miscellaneous test conditions such as end restraint, initial one-dimensional stress, isotropic consolidation stress and load increment duration in the stress-strain behaviour and strength characteristics were studied in detail and will be published elsewhere. Special precautions were taken to reduce the effects of nonuniformity in deformation by the use of lubricated ends (Rowe and Barden, 1964) for the samples. Leakage was virtually eliminated by the use of silicone oil (Ting, 1968).

STRESS STRAIN THEORY OF ROSCOE AND POOROOSHASB BASED ON UNIQUE (*p, q, e*) SURFACE FOR NORMALLY AND LIGHTLY OVERCONSOLIDATED CLAYS

Assuming a unique (*p, q, e*) surface, an incremental stress strain theory was proposed by Roscoe and Poorooshasb (1963) in the form

$$d\epsilon_1 = \left(\frac{d\epsilon_1}{d\eta}\right)_v d\eta + \left(\frac{d\epsilon_1}{dv}\right)_\eta dv \tag{1}$$

where $(d\epsilon_1/d\eta)_v$ corresponds to the variation of ϵ_1 with η in an undrained test and $(d\epsilon_1/dv)_\eta$ represents the variation of ϵ_1 with v in a constant η stress path. The total strain experienced at any instant for any state path on the state boundary surface *ABCD* in Fig 1 is calculated by integrating (1). η is defined as the stress ratio q/p.

THE CRITICAL STATE CONCEPT AND THE ASSOCIATED STRESS STRAIN THEORIES

Roscoe, Schofield and Wroth (1958) put forward a basic concept that the end points of all tests performed on remoulded saturated clay lie on a unique line in the (*p, q, e*) space. This line was subsequently defined as the critical state line. At the critical state, the material is capable of having unlimited shear distortion without any further change in stresses or volume. The critical state line is assumed (i) to have a constant slope *M* in the (*q, p*) space and (ii) to be parallel to the virgin compression line in the (*e*, log *p*) plot. These statements about the critical state line were studied in detail in James and Balasubramaniam (1971 (a) and (b)). The parameter *M* is assumed to be a fundamental property of the material, it was used by Roscoe, Schofield and Thurairajah (1963) in deriving an expression for the energy dissipated in plastic distortion. A stress-strain theory was proposed by the authors based on an energy equation and some of the concepts of the theories of plasticity as developed for metals. This stress-strain theory was subsequently presented in a concise form and the ideal material which satisfies all the assumptions made was called the Cam-Clay (see Roscoe and Schofield (1963). The expression for energy dissipation was later modified by Roscoe and Burland (1968) and a modified theory was presented in the light of the new energy equation. Wroth (1965) using the energy equation of Roscoe, Schofield and Thurairajah (1963) proposed an independent theory based on an empirical strain equation

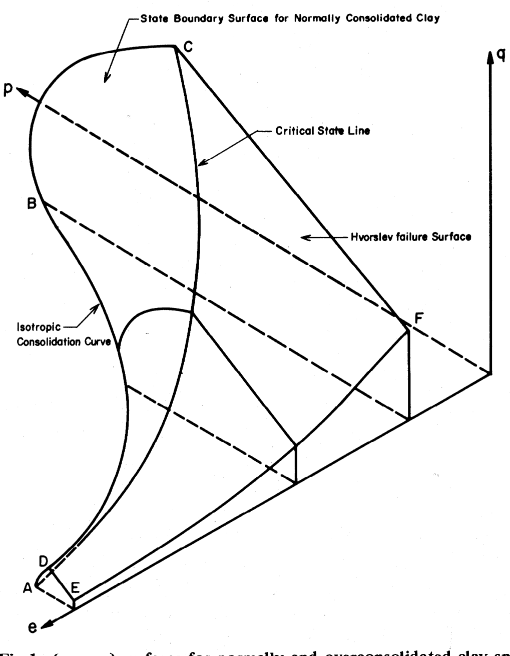

Fig 1. (*p, q, e*) surfaces for normally and overconsolidated clay specimens (after Roscoe, Schofield and Wroth, 1958).

which does not depend on theories of plasticity. This theory predicts identical strains to those obtained from the Cam-Clay theory for normally consolidated clays.

INCREMENTAL STRAIN EQUATION FOR CAM-CLAY THEORY AND THE MODIFIED THEORY

The application of the concepts of plasticity for the deformation of clays has been discussed in detail in Balasubramaniam (1969). The incremental strains as predicted from the Cam-Clay theory are as follows

$$dv = \frac{1}{p}\left(\frac{\lambda}{1+e}\right)\left[\frac{(1-\frac{k}{\lambda})}{M}(dq - \eta dp) + dp\right] \tag{2}$$

$$d\epsilon = \left(\frac{\lambda - k}{1+e}\right)\left(\frac{1}{M-\eta}\right)\left[1 + \frac{1}{M}\left(\frac{dq}{dp} - q/p\right)\right]\frac{dp}{p} \tag{3}$$

The corresponding expressions for the Modified theory are

$$dv = \left(\frac{\lambda}{1+e}\right)\left[\frac{dp}{p} + \left(\frac{2\eta}{M^2+\eta^2}\right)\left(1 - \frac{k}{\lambda}\right)d\eta\right] \tag{4}$$

$$d\epsilon = \left(\frac{\lambda - k}{1+e}\right)\left(\frac{2\eta}{M^2-\eta^2}\right)\left[\frac{dp}{p} + \left(\frac{2\eta}{M^2+\eta^2}\right)d\eta\right] \tag{5}$$

The incremental stress-strain relationships given by (2) to (5) are only dependent on three fundamental soil constants *M*, λ and *k*. Where *M* is the slope of the critical state line in the (*q, p*) plot, λ is the slope of the isotropic consolidation line in an (*e*, log *p*) plot, *k* is the slope of the isotropic swelling line in an (*e*, log *p*) plot, *e* is the voids ratio and $\eta = q/p$ the stress ratio. Also for Cam-Clay theory

$$\frac{dv}{d\epsilon} = \left(\frac{\lambda}{\lambda - k}\right)(M - \eta) \tag{6}$$

and for Modified theory

$$\frac{dv}{d\epsilon} = \left(\frac{\lambda}{\lambda - k}\right)\left(\frac{M^2 - \eta^2}{2\eta}\right) \tag{7}$$

The flow rule based on the incremental stress-strain theory of Roscoe and Poorooshasb

The incremental stress strain theory of Roscoe and Poorooshasb presented in (1) will now be presented in a slightly different form to facilitate comparison with the other theories. The basic equation is

$$(d\epsilon)_{\text{drained}} = (d\epsilon)_{\text{undrained}} + \left(\frac{d\epsilon}{dv}\right)_\eta dv \tag{8}$$

If it is assumed that

(i) for stress paths starting from a given point on the state boundary surface and provided the subsequent stress paths lie throughout on the state boundary surface then v is a unique function of η and p. Expressed mathematically this assumption is

$$v = F(\eta, p) \tag{9}$$

(ii) the shear strain in an undrained test is a continuous and differentiable function of the stress ratio

$$(\epsilon)_{\text{undrained}} = \int_0^\eta f_1(\eta)\, d\eta \tag{10}$$

(iii) the slope $\left(\frac{dv}{d\epsilon}\right)_\eta$ in the (v, ϵ) space of the anisotropic consolidation path is only a function of the stress ratio

$$\text{ie} \quad \left(\frac{d\epsilon}{dv}\right)_\eta = f_2(\eta) \tag{11}$$

then (8) can be expressed as

$$d\epsilon = f_1(\eta)d\eta + f_2(\eta)dF(\eta, p) \tag{12}$$

Also from (9)

$$dv = \frac{\partial F}{\partial \eta}d\eta + \frac{\partial F}{\partial p}dp \tag{13}$$

In the case of a constant p stress path the second term in (13) becomes zero and therefore $\partial F/\partial\eta$ is given by the slope of the (v, η) relationship observed during a constant p test. Similarly in the case of an anisotropic consolidation stress path the first term in (13) becomes zero and $\partial F/\partial p$ is given by the slope of the (v, p) relationship for the value of η corresponding to the particular anisotropic consolidation path under consideration. This slope is found to be independent of η and is $\dfrac{\lambda}{1+e}$. From (12) and (13)

$$\frac{dv}{d\epsilon} = \frac{\left(\dfrac{\partial F}{\partial \eta} d\eta + \dfrac{\partial F}{\partial p} dp\right)}{\left[f_1(\eta)d\eta + f_2(\eta)\left(\dfrac{\partial F}{\partial \eta} d\eta + \dfrac{\partial F}{\partial p} dp\right)\right]} \tag{14}$$

Equation (14) provides the relevant expressions for the ratio of the total strain increments in terms of the functions $f_1(\eta)$, $f_2(\eta)$ and $F(\eta, p)$ and the differentials $d\eta$ and dp.

It will be shown subsequently that (12), (13) and (14) are valid for any stress path for which the stress increment direction lies between the local undrained stress path and the local anisotropic consolidation path. The functions $f_1(\eta)$, $f_2(\eta)$ and $F(\eta, p)$ are determined directly from the experimental observations of a single undrained test and a series of anisotropic consolidation tests.

Since the above theory of Roscoe and Poorooshasb gives accurate strain predictions (and is essentially based on experimental observation) it will be of interest to rearrange the terms in the equations (for the prediction of strains) of the Cam-Clay theory and the Modified theory to obtain the equivalents of $f_1(\eta)$, $f_2(\eta)$ and $F(\eta, p)$. This will be carried out in the following section.

The conditions under which the Cam–Clay theory and its modification will predict identical strains to those of the stress strain theory of Roscoe and Poorooshasb

For the Cam-Clay theory

$$dv = \left(\frac{\lambda}{1+e}\right)\frac{dp}{p} + \left(\frac{\lambda - k}{1+e}\right)\frac{1}{M}d\eta \tag{15}$$

i.e.

$$dv = \frac{\partial F^*}{\partial p}dp + \frac{\partial F^*}{\partial \eta}d\eta \tag{16}$$

where

$$\frac{\partial F^*}{\partial p} = \left(\frac{\lambda}{1+e}\right)\frac{1}{p} \tag{17}$$

and

$$\frac{\partial F^*}{\partial \eta} = \frac{1}{M}\left(\frac{\lambda - k}{1+e}\right) \tag{18}$$

From (3)

$$d\epsilon = \left(\frac{\lambda - k}{1+e}\right)\frac{1}{M}\,\frac{d\eta}{(M-\eta)} + \left(\frac{\lambda - k}{1+e}\right)\frac{1}{(M-\eta)}\,\frac{dp}{p} \tag{19}$$

i.e.

$$d\epsilon = f_1^*(\eta)d\eta + f_2^*(\eta)dF^*(\eta, p) \tag{20}$$

where

$$f_1^*(\eta) = \left(\frac{\lambda - k}{1+e}\right)\frac{1}{M}\left(\frac{1}{M-\eta}\right)\frac{k}{\lambda} \tag{21}$$

and

$$f_2^*(\eta) = \left(\frac{\lambda - k}{\lambda}\right)\left(\frac{1}{M-\eta}\right) \tag{22}$$

From (18), (21) and (22) it can be shown that

$$f_1^*(\eta) = \left(\frac{k}{\lambda - k}\right)f_2^*(\eta)\frac{\partial F^*}{\partial \eta} \tag{23}$$

These relationships will be discussed further after deriving, the corresponding relations for the Modified theory. Following the same procedure as that adopted for the Cam-Clay theory and using the superfix** for the expressions in the Modified theory it can be shown that

$$\frac{\partial F^{**}}{\partial \eta} = \left(\frac{\lambda - k}{1+e}\right)\left(\frac{2\eta}{M^2+\eta^2}\right) \tag{24}$$

$$\frac{\partial F^{**}}{\partial p} = \left(\frac{\lambda}{1+e}\right)\frac{1}{p} \tag{25}$$

$$f_2^{**}(\eta) = \left(\frac{\lambda - k}{\lambda}\right)\left(\frac{2\eta}{M^2-\eta^2}\right) \tag{26}$$

$$f_1^{**}(\eta) = \left(\frac{\lambda - k}{1+e}\right)\left(\frac{2\eta}{M^2-\eta^2}\right)\left(\frac{2\eta}{M^2+\eta^2}\right)\frac{k}{\lambda} \tag{27}$$

$$f_1^{**}(\eta) = f_2^{**}(\eta)\left(\frac{k}{1+e}\right)\left(\frac{2\eta}{M^2+\eta^2}\right) \tag{28}$$

$$f_1^{**}(\eta) = \left(\frac{k}{\lambda - k}\right)f_2^{**}(\eta)\frac{\partial F^{**}}{\partial \eta} \tag{29}$$

Comparing (29) for the Modified theory with (23) for the Cam-Clay theory it is observed that the two equations are similar. It can be shown from (24) and (26) that

$$\frac{\partial F^{**}}{\partial \eta} = \frac{\left(1 - \dfrac{k}{\lambda}\right)\left(\dfrac{\lambda}{1+e}\right)\dfrac{1}{M}}{\left[1 + \left(\dfrac{1-\dfrac{k}{\lambda}}{M}\right)^2\left(\dfrac{1}{f_2^{**}(\eta)}\right)^2\right]^{1/2}} \tag{30}$$

which can be arranged to give

$$f_2^{**}(\eta) = \frac{\dfrac{\partial F^{**}}{\partial \eta}}{\left[\left(\dfrac{\lambda}{1+e}\right)^2 - \left(\dfrac{M}{1-\dfrac{k}{\lambda}}\right)^2\left(\dfrac{\partial F^{**}}{\partial \eta}\right)^2\right]^{1/2}} \tag{31}$$

The function $\dfrac{\partial F^{**}}{\partial \eta}$ is the slope of the (v, η) characteristic of a constant p test, the function $f_2^{**}(\eta)$ is the slope of the anisotropic consolidation path in the (v, ϵ) space. Hence (31) indicates that there is a fundamental relation existing between the slope of the anisotropic consolidation line in the (v, ϵ) space and the slope of the constant p test path in the (v, η) space. Consequently in the calculation of the shear strains caused by any imposed stress increment using the incremental stress-strain theory of Roscoe and Poorooshasb (1963) it is possible to obtain the magnitude of $(dv/d\epsilon)_\eta$ from a single constant p test. In other words it is not necessary to carry out anisotropic consolidation tests to find out $f_2(\eta)$. In the stress-strain theory of Roscoe and Poorooshasb, the function F can be obtained from results of one undrained test, provided the value of λ is obtained during the isotropic consolidation phase of the sample preparation. The function F, representing the state boundary surface is therefore obtained from experimental observations. On the other hand the function F^* and F^{**} for the state boundary surface of the Cam-Clay theory and its Modification come from the concept of normality and the respective energy equations. Hence for either the Cam-Clay theory or the Modified theory to predict identical strains (volumetric) to those of the stress strain theory of Roscoe and Poorooshasb (1963) it is clear that the conditions

$$F^* = F \quad \text{(Cam-Clay theory)} \tag{32a}$$

$$F^{**} = F \quad \text{(Modified theory)} \tag{32b}$$

should be satisfied.

A further condition will be required if the shear strains as predicted from either of the two theories are to be identical to those obtained from the Roscoe and Poorooshasb theory and therefore to agree with those experimentally observed.

The general expression for the incremental shear strain in the Roscoe and Poorooshasb theory is

$$d\epsilon = f_1(\eta)d\eta + f_2(\eta)dF \tag{33}$$

Since the shear strains predicted from the Cam-Clay theory and the Modified theory are to be identical to those of Roscoe and Poorooshasb theory for all imposed stress paths they must be relevant for an undrained path, so that

$$f_1(\eta) = f_1^*(\eta) = f_1^{**}(\eta) \tag{34}$$

and also for a constant η path so that

$$f_2(\eta) = f_2^*(\eta) = f_2^{**}(\eta) \tag{35}$$

It has already been shown in (32a) and (32b) that identical volumetric strains in all three theories can only be obtained if

$$F = F^* = F^{**} \tag{36}$$

and hence the whole problem reduces to satisfying (32a), (32b), (34) and (35). It should be recalled that $f_1(\eta)$ is the experimentally observed value of the shear strain in an undrained test, and it can be seen from (23) and (29), that if

$$f_2^*(\eta) = f_2^{**}(\eta) \text{ and } F^* = F^{**} \text{ then}$$

$$f_1^*(\eta) = f_1^{**}(\eta) \tag{37}$$

The energy equations used in the Cam-Clay theory and the Modified theory determine the functions $f_2^*(\eta)$ and $f_2^{**}(\eta)$ respectively, and also F^* and F^{**}. They therefore also fix $f_1^*(\eta)$ and $f_1^{**}(\eta)$ respectively. In the Roscoe and Poorooshasb theory the functions $f_1(\eta), f_2(\eta)$ and F are independent whereas the corresponding functions for either of the other two theories are interrelated. Consequently if the energy equation of either of these two theories does not predict F correctly, then it will not predict $f_1(\eta)$ or $f_2(\eta)$ correctly.

The concept of an extra yield locus for shear distortion without plastic volume change

Replotting the undrained test results of Loudon (1967) for lightly overconsolidated clays in a (p, q) plot, Roscoe and Burland (1968) showed that for specimens of overconsolidated ratio 1 to 2.2 the contours of constant shear strain beneath the state boundary surface coincide with the contours of constant q. They further stated that for normally and lightly overconsolidated clays the contours of constant q can be considered as a series of yield loci. These constant q yield loci extend up to the conventional yield loci used in the Modified theory, Roscoe and Burland (1968) named this latter yield locus the volumetric yield locus. Thus for stress increments directed outside the volumetric yield locus the stress point is assumed to move with the intersection of the constant q yield locus and its corresponding volumetric yield locus. However for stress increments directed inside the volumetric yield locus in a direction in which q is increasing the stress point is assumed to move through a series of constant q yield locus. Based on this new concept, Roscoe and Burland (1968) proposed revised flow rules. A brief summary of these revised flow rules which lead to what Roscoe and Burland (1968) called their Revised theory is presented in Balasubramaniam (1969). For the remainder of this paper this theory will be called the Revised theory. The values of λ, k and M are taken to be 0.26, 0.06 and 0.9 respectively.

The shear strain associated with the additional constant q yield locus

In Fig 2 for any particular value of the deviator stress, let η be the magnitude of the stress ratio, at which the constant q line meets the volumetric yield locus. Then the contribution to the shear strain from the constant q yield locus can be expressed as

$$\bar{\epsilon} = \phi(\eta) \tag{38}$$

Fig 2(a) illustrates the variation of $\bar{\epsilon}$ with the stress ratio q/p. Differentiating,

$$d\bar{\epsilon} = \frac{\partial\phi}{\partial\eta}d\eta = \phi'(\eta)d\eta \tag{39}$$

where

$$\phi'(\eta) = \frac{\partial\phi}{\partial\eta}$$

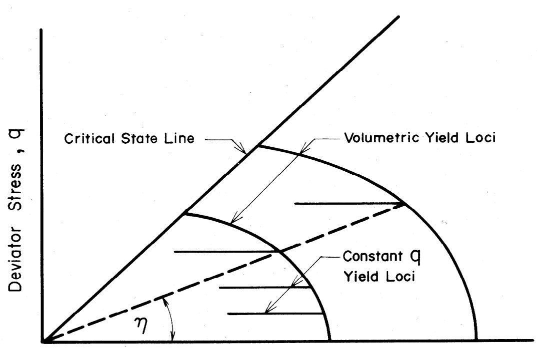

Fig 2. Volumetric and constant q yield loci of Roscoe and Burland (1968).

The revised flow rule of Roscoe and Burland (1968)

The following relationships were used by Roscoe and Burland (1968) for the determination of the incremental strains corresponding to the stress increments applied to a specimen such that η increases with the stress state lying on the state boundary surface.

$$dv = \frac{\lambda}{1+e}\left[\frac{dp}{p} + \left(1 - \frac{k}{\lambda}\right)\left(\frac{2\eta}{M^2+\eta^2}\right)d\eta\right] \tag{4}$$

and

$$d\epsilon = \left(\frac{\lambda-k}{1+e}\right)\left(\frac{2\eta}{M^2-\eta^2}\right)\left[\frac{dp}{p} + \left(\frac{2\eta}{M^2+\eta^2}\right)d\eta\right] + \phi'(\eta)\,d\eta \tag{40}$$

Comparing the above set of equations with the incremental stress-strain theory of Roscoe and Poorooshasb it is evident that for stress increments applied in the range in which η increases (where the applied stress paths lie between the undrained stress path and the anisotropic consolidation path in the (p, q) space the Revised theory will predict identical strain if and only if

$$f_1(\eta) = f_1^{**}(\eta) + \phi'(\eta) \tag{41}$$

If condition (41) is satisfied then the Revised theory may be put in the form of Roscoe and Poorooshasb theory

$$dv = \frac{\partial F^{**}}{\partial p}dp + \frac{\partial F^{**}}{\partial\eta}d\eta \tag{42}$$

and

$$d\epsilon = \left[f_1^{**}(\eta) + \phi'(\eta)\right]d\eta + f_2^{**}(\eta)\,dF^{**} \tag{43}$$

where

$$f_2^{**}(\eta) = \frac{\dfrac{\partial F^{**}}{\partial\eta}}{\left[\left(\dfrac{\lambda}{1+e}\right)^2 - \left(\dfrac{M}{1-\dfrac{k}{\lambda}}\right)^2\left(\dfrac{\partial F^{**}}{\partial\eta}\right)^2\right]^{1/2}} \tag{31}$$

and

$$f_1^{**}(\eta) = \left(\frac{k}{\lambda-k}\right)f_2^{**}(\eta)\frac{\partial F^{**}}{\partial\eta} \tag{29}$$

The concept of normality and the energy balance used by Roscoe and Burland provides a possible method of obtaining the function F^{**}, which satisfies (29), (31) and (42). Also the concept of the constant q yield loci provides a method of determining $\phi(\eta)$.

Experimental observations will now be provided to illustrate that the function F^{**} and $\phi(\eta)$ can be used to predict strains for all stress paths, lying in between the undrained and anisotropic consolidation paths.

The experimentally observed form of $v = F(\eta, p)$

In Fig 3 the volumetric strain contours are plotted from a variety of drained tests each with different values of (dq/dp), the zero volumetric strain contour from an undrained test path (T_3) is also shown. The

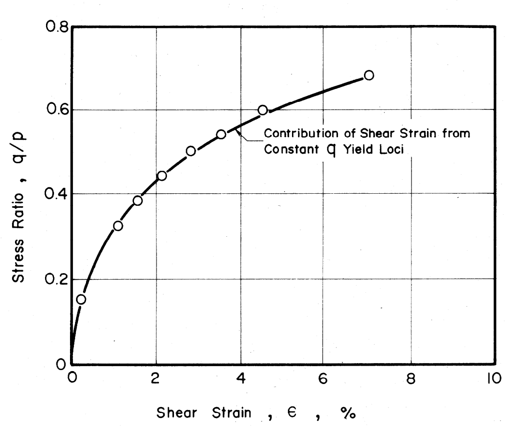

Fig 2(a). Contribution of shear strain from constant q yield loci.

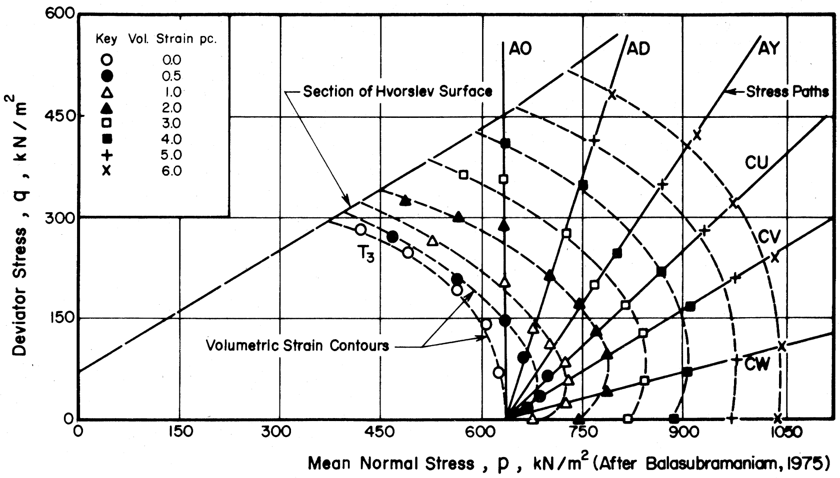

Fig 3. Volumetric strain contours for normally consolidated specimens of Kaolin.

stress paths are indicated by the full lines. From the volumetric strain contours (shown dotted) it is evident that except for stresses close to the isotropic stress (from which the specimens are sheared) the contours are similar to that obtained from the undrained stress path. All the specimens were sheared from 621.3 kNm^{-2} isotropic stress. Fig 3 illustrates that a state surface in (p, q, v) space exists for all specimens sheared from 621.3 kNm^{-2} isotropic stress with an increasing stress ratio. Balasubramaniam (1969) has shown that except for stress paths with directions lying inside the tangent to the current volumetric strain contours plotted in Fig 3, for all other stresses, the volumetric strain contours plotted in Fig 3 are unique. This unique surface will now be used to calculate the shear strains experienced by specimens in drained tests similar to those shown in Fig 3, using the hypothesis of Roscoe and Poorooshasb (1963).

The $(dv/d\epsilon)_\eta$ of anisotropic consolidation paths for samples initially isotropically consolidated to 621.3 kNm^{-2}

In this section the experimental observations on the compression tests with constant stress ratio (anisotropic consolidation) are presented. The author's results confirm the experimental finding of Roscoe and Poorooshasb (1963), Thurairajah (1961) that during the anisotropic consolidation (with constant stress ratio η) the ratio of the incremental strains, namely, $dv/d\epsilon$, is a constant and is dependent on the stress ratio η.

The points in Fig 4 show the variation of $(dv/d\epsilon)_\eta$ corresponding to five different values of stress ratio η as observed experimentally. The curve in Fig 4 gives the predicted variation of $(dv/d\epsilon)_\eta$ with η as calculated from the expressions

$$\frac{1}{f_2^*(\eta)} = \left(\frac{dv}{d\epsilon}\right)_\eta = \left(\frac{\lambda}{\lambda - k}\right)(M - \eta) \tag{22}$$

and

$$\frac{1}{f_2^{**}(\eta)} = \left(\frac{dv}{d\epsilon}\right)_\eta = \left(\frac{\lambda}{\lambda - k}\right)\left(\frac{M^2 - \eta^2}{2\eta}\right) \tag{26}$$

It can be seen that the variation of $f_2^{**}(\eta)$ as calculated from (26) (Modified theory) is in excellent agreement with the experimental observations which give $f_2(\eta)$. Therefore $f_2(\eta) = f_2^{**}(\eta)$. However for $\eta < 0.5$ it is evident that $f_2^*(\eta)$, for Cam-Clay theory, is much larger than $f_2(\eta)$. Hence over most of the range of η from 0 to M

$$f_2(\eta) \neq f_2^*(\eta)$$

The incremental volumetric strain during anisotropic consolidation

The changes in voids ratio with log p during anisotropic consolidation tests at three different values of η are found to be linear with the same slope. Fig 5 illustrates the variation for one particular case. Hence during anisotropic consolidation

$$dv = \left(-\frac{\lambda}{1+e}\right)\frac{dp}{p} \tag{44}$$

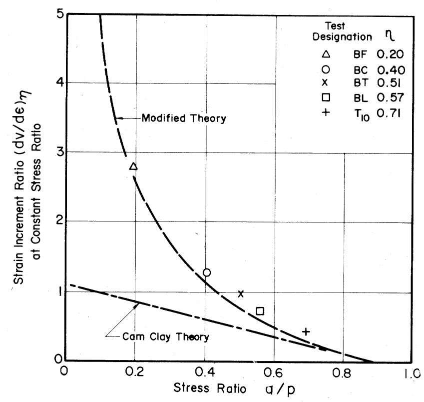

Fig 4. Observed and predicted $\left[\left(\frac{dv}{d\epsilon}\right)_\eta, \eta\right]$ relationship for anisotropic consolidation tests.

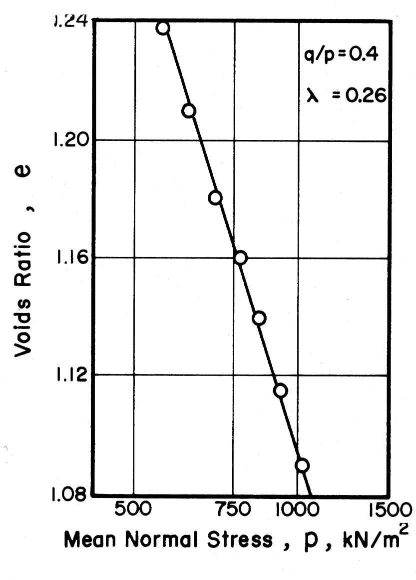

Fig 5. The $(e, \log p)$ characteristic during anisotropic consolidation.

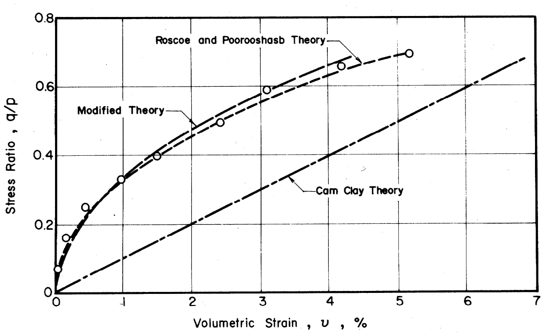

Fig 6. Volumetric strains in constant p test.

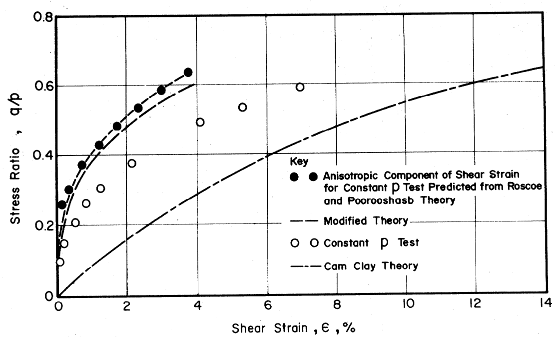

Fig 8. Observed and predicted shear strains for constant p test.

From (13), (17), (25) and (44).

$$\frac{\partial F}{\partial p} = \frac{\partial F^*}{\partial p} = \frac{\partial F^{**}}{\partial p} = \left(-\frac{\lambda}{1+e}\right)\frac{1}{p}$$

In other words all three theories predict correctly the volumetric strains observed experimentally during anisotropic consolidation tests.

Volumetric strains in constant p tests

The volumetric strains observed during a constant p test at 621.3 kNm^{-2} are represented by the points in Fig 6. The curves in Fig 6 refer to the predicted values from various stress-strain theories. It is evident that the Modified theory and the incremental stress-strain theory of Roscoe and Poorooshasb closely predicts the strains in a constant p test. The Cam-Clay theory overpredicts the strains in constant p tests.

Hence is is evident that

$$\frac{\partial F}{\partial \eta} = \frac{\partial F^{**}}{\partial \eta}$$

and

$$\frac{\partial F}{\partial \eta} \neq \frac{\partial F^*}{\partial \eta}$$

Shear strains in constant p tests

The shear strains observed in a constant p test and in an undrained test are illustrated in Fig 7. In the same figure the predicted components of shear strain due to the anisotropic consolidation part of a constant p test is also indicated. The two components of shear strain are now added at each stress ratio, to give the Roscoe and Poorshasb prediction. Excellent agreement is obtained between this prediction and the experimentally observed strains.

In Fig 8 the relationship between η and ϵ is shown for (i) the anisotropic component of the shear strains in a constant p test as calculated from the stress-strain theory of Roscoe and Poorooshasb (ii) the shear strains predicted from the Modified theory (iii) the experimentally observed shear strains in a constant p test and (iv) the strains predicted from the Cam-Clay theory. From this figure the following observations may be made

(a) the Modified theory underpredicts the experimentally observed shear strain, but agrees with the anisotropic component of the constant p test shear strains, as calculated from the stress-strain theory of Roscoe and Poorooshasb.

(b) the Cam-Clay theory overpredicts the shear strain in a constant p test.

Therefore it can be concluded that

$$f_1^{**}(\eta)\, d\eta + f_2^{**}(\eta)\, dF^{**} = f_2(\eta)\, dF \tag{45}$$

and

$$f_1^{*}(\eta)\, d\eta + f_2^{*}(\eta)\, dF^{*} \gg f_1(\eta)\, d\eta + f_2(\eta)\, dF \tag{46}$$

Having seen that the Modified theory predicts approximately the same amount of the shear strain as the anisotropic component, it is evident that though $f_1^{**}(\eta)\, d\eta$ and $f_2^{**}(\eta)\, dF^{**}$ can be independently derived from the Modified theory and considered to be the undrained shear strain and the anisotropic component of the shear strain, it is in fact not so in actual practice and both the sum of $f_1^{**}(\eta)\, d\eta$ and $f_2^{**}(\eta)\, dF^{**}$ is equal to the anisotropic component of the shear strain. If it is recalled that both $f_1^{**}(\eta)$ and $f_2^{**}(\eta)$ are functions of F^{**} as given by (29) and (31) it is absolutely clear that a single function of the form F^{**} could not be used for the prediction of shear strain in any test other than that of the anisotropic consolidation test. The energy equation used in the Modified theory is only a means of obtaining the function F^{**} for the state boundary surface and the selection of a wrong energy equation would result in a wrong F^{**} and hence the volumetric strain and the anisotropic component of the shear strain as predicted from this F^{**} would be in error.

To the anisotropic component of the shear strain as illustrated in Fig 8 is added the undrained component of the shear strain as predicted by the Revised theory. The result is presented in Fig 9 together with experimentally observed strains. Excellent agreement is noted between the experimentally observed strains and the theoretical strains.

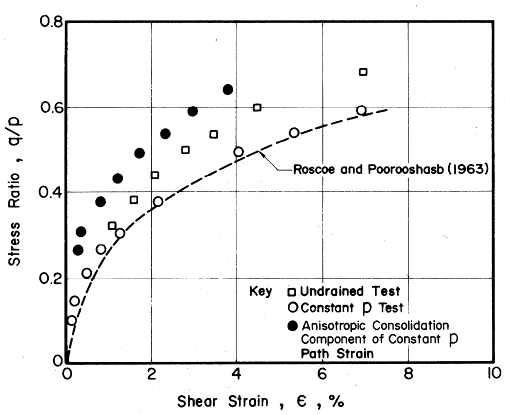

Fig 7. Shear strains in undrained test and constant p test.

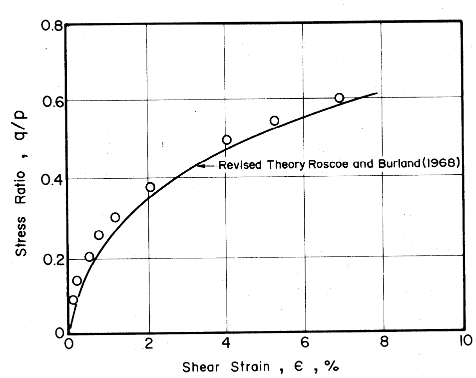

Fig 9. Shear strains predicted from "Revised Theory" for constant p test.

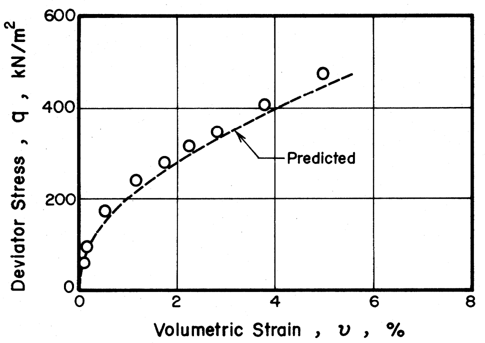

Fig 10.. Observed and predicted volumetric strain for constant p test.

It is therefore concluded that

$$f_1^{**}(\eta)\,d\eta + f_2^{**}(\eta)dF^{**} + \phi'(\eta)\,d\eta = f_1(\eta)\,d\eta + f_2(\eta)\,dF \quad (47)$$

where

$$\phi'(\eta)\,d\eta = f_1(\eta)\,d\eta$$

The above results indicate that the Modified theory in its own form lacks the undrained component of shear strain used by Roscoe and Poorooshasb (1963) in the calculation of shear strains in any test with $\eta > 0$ and $d\eta > 0$. Once this component is added to the shear strains predicted by the Modified theory, the Revised values are in close agreement with the strains predicted from the incremental stress-strain theory of Roscoe and Poorooshasb (1963).

Strains predicted by the stress-strain theory of Roscoe and Poorooshasb

Balasubramaniam (1969) has studied in detail the effects of history and other factors on the state boundary surface in the $(q/p_e, p/p_e)$ plot, The topics discussed include the effects of (a) the initial moisture content of the slurry from which the specimens were prepared (b) the one dimensional stress used in the sample preparation (c) different levels of isotropic stress (d) the end restraint, the sample size and load increment duration and (e) the duration of the undrained and fully drained tests. These factors are found to influence the state boundary surface and methods which take into account these effects in the stress-strain theories are still to be developed. At present the author confines himself to samples which have been prepared under identical conditions and tested under the same conditions at 621.3 kNm^{-2} isotropic stress. At this stress level the state boundary surface is unique and is independent of the applied stress paths as indicated in Fig 3. Also the sections of this surface by constant voids ratio planes are geometrically similar to the undrained stress path in the (q, p) space. The predicted and experimentally observed strains in three types of tests (constant p test, constant cell pressure and a drained test with applied stress path of slope less than three) are presented in Figs 10 to 15. Excellent agree-

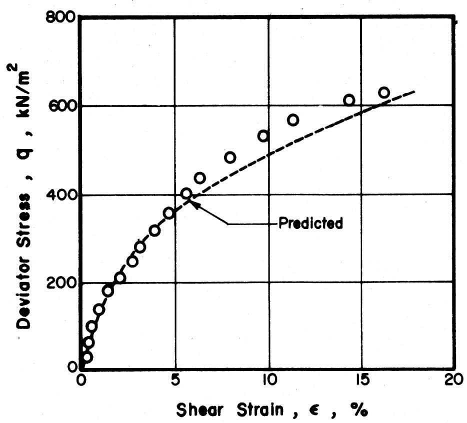

Fig 13. Observed and predicted shear strain for fully drained test.

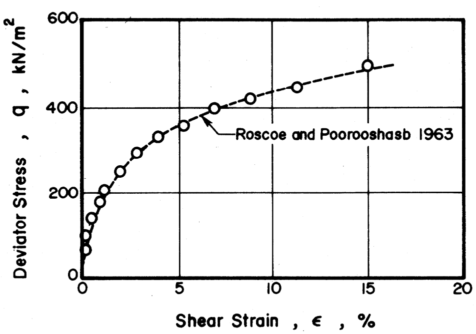

Fig 11. Observed and predicted shear strain for constant p test.

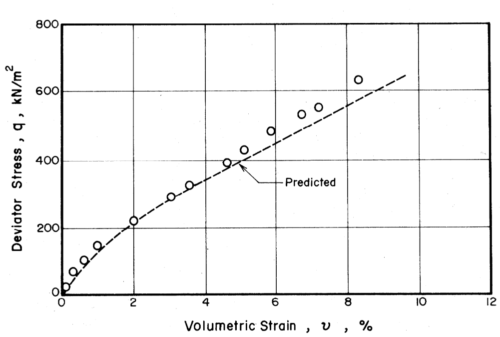

Fig 12. Observed and predicted volumetric strain for fully drained test.

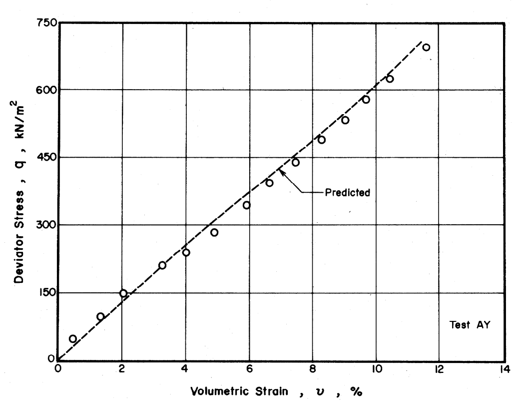

Fig 14. Observed and predicted volumetric strain during drained test with applied stress path of slope (= dq/dp) of 1.5.

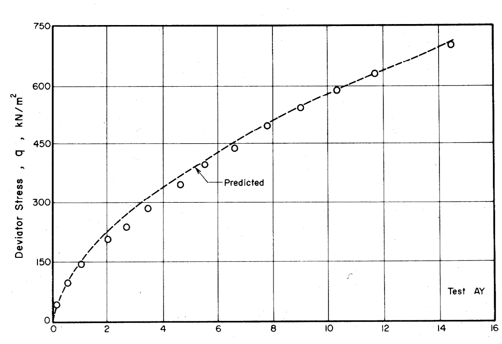

Fig 15. Observed and predicted shear strain during drained test with applied stress path of slope (= dq/dp) of 1.5.

ment is obtained between the predicted strains and the experimentally observed strains for all three types of tests. The stress paths considered so far are such that the stress ratio η always increases and for these paths the Roscoe and Poorooshasb theory and the Revised theory are found to predict identical strains to those of the experimentally observed strains.

CONCLUSIONS

The stress-strain theory of Roscoe and Poorooshasb was presented in a modified form to enable it to be compared with the other theories developed at Cambridge based on an energy balance and the concepts of plasticity. Conditions were stated and verified experimentally to illustrate when the Cam-Clay theory or the Modified theory or the Revised theory would predict identical strains to those of the Roscoe and Poorooshasb theory. This investigation showed that the basic philosophy between all four theories is similar and hence they could be expressed mathematically in a similar form. The energy balance and the normality condition used together is only a mean for obtaining this mathematical similarity and the particular combination of the energy balance used in the Revised theory together with the normality relation yields a method by which identical volumetric strains to those of the Roscoe and Poorooshasb theory could be obtained. The shear strain predicted by the use of the above concept is only equivalent to the anisotropic component of the shear strain in a drained test and hence the additional constant q yield locus used in the Revised theory is a necessity (to satisfy the undrained component of shear strain) for the successful prediction of shear strains.

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