Critical state soil mechanics in prediction of strains

Elements of Critical State Soil Mechanics in Prediction of Strains

1. Definition of stress and strain parameters

2. State parameters and the use of (q, p, e) and applied stress paths and effective stress paths

3. Normally consolidated and over-consolidated states

4. Stress strain behavior and strength

Elements of Critical State Soil Mechanics in Prediction of Strains (contd.)

5. Elementary concepts in theory of elasticity and plasticity

6. Theories for normally consolidated clays

7. Natural deposits and over-consolidated states

8. Numerical analysis

Classical contributions

1. Rendulic (1936)
   (on constant voids ratio contours)

2. Hvorslev (1936)
   (on mean equivalent pressure)

3. Roscoe, Schofield & Wroth (1958)
   (on state boundary surface)

4. Drucker (1959)
   (Stability criterion

5. Poorooshasb (1961) and
Roscoe & Poorooshasb (1963)
(Incremental drained strain as incremental undrained and anisotropic consolidation strains)

6. Thurairajah (1961)
(Energy Balance equation)
Calladine (1963)
(on elastic wall concept)
Roscoe, Schofield &Thurairajah (1963)

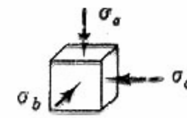
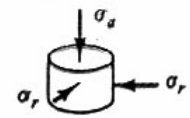
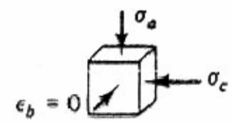
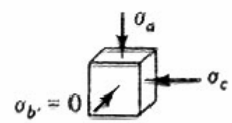
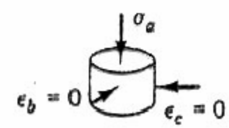
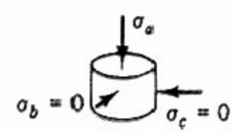
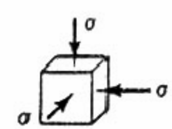
7. **Burland (1965)**
Roscoe & Burland (1967)
(Modified energy balance equation & constant q yield loci)

8. **Wroth & Loudon (1967)**
Pender (1970)
(on modeling over-consolidated clays)

Monday-5

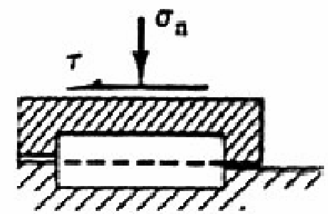
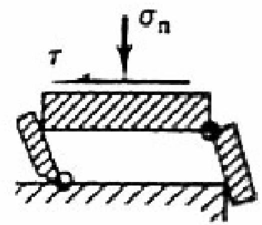
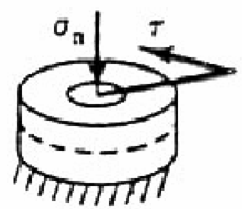
1 Stresses and Strains

Table 1.1 Stress and strain conditions on equipment with Principal stress and strain conditions (From Atkinson & Bransby)

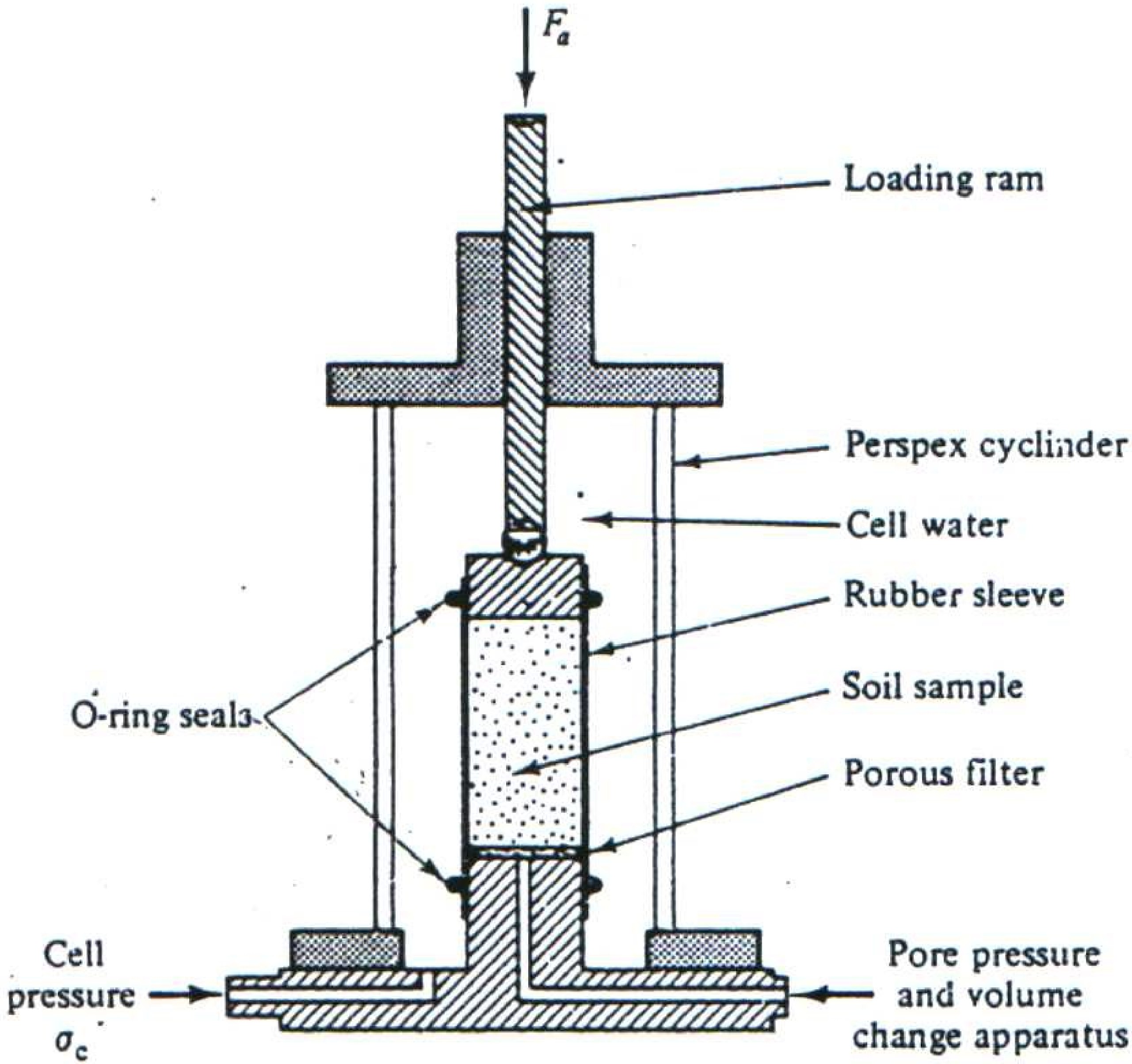
Special conditions	Name of test	Diagram
$\sigma_a \neq \sigma_b \neq \sigma_c$	True triaxial	
$\sigma_b = \sigma_c = \sigma_r$	Cylindrical compression The 'triaxial' test	
$\epsilon_b = 0$	Plane strain or biaxial	
$\sigma_b = 0$	Plane stress	
$\epsilon_b = \epsilon_c = \epsilon_r = 0$	One-dimensional compression The oedometer test	
$\sigma_b = \sigma_c = \sigma_r = 0$	Uniaxial compression or unconfined compression	
$\sigma_a = \sigma_b = \sigma_c = \sigma$	Isotropic compression	

Stress conditions in some common soil tests

Table 1.2: Equipment in which the boundary stresses and strains are not under principal conditions from (Atkinson & Bransby).

Platens	Name of test	Diagram
Rough: non-rotating	Direct shear The shear box test	
Rough: rotating	Simple shear	
Rough: non-rotating	Torsion The ring shear test	

Shear tests for soils



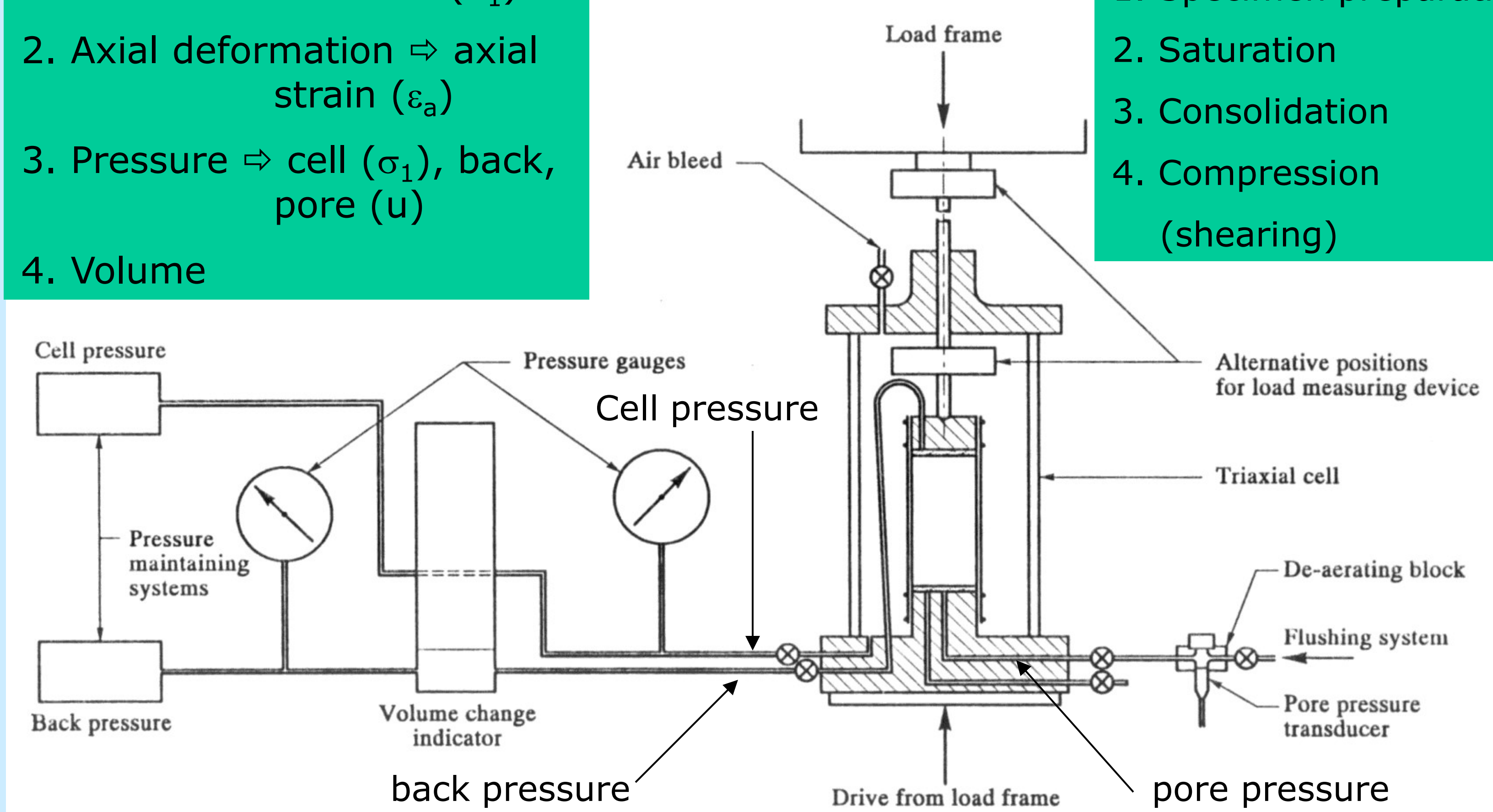
The triaxial apparatus

Shear Strength of Soils

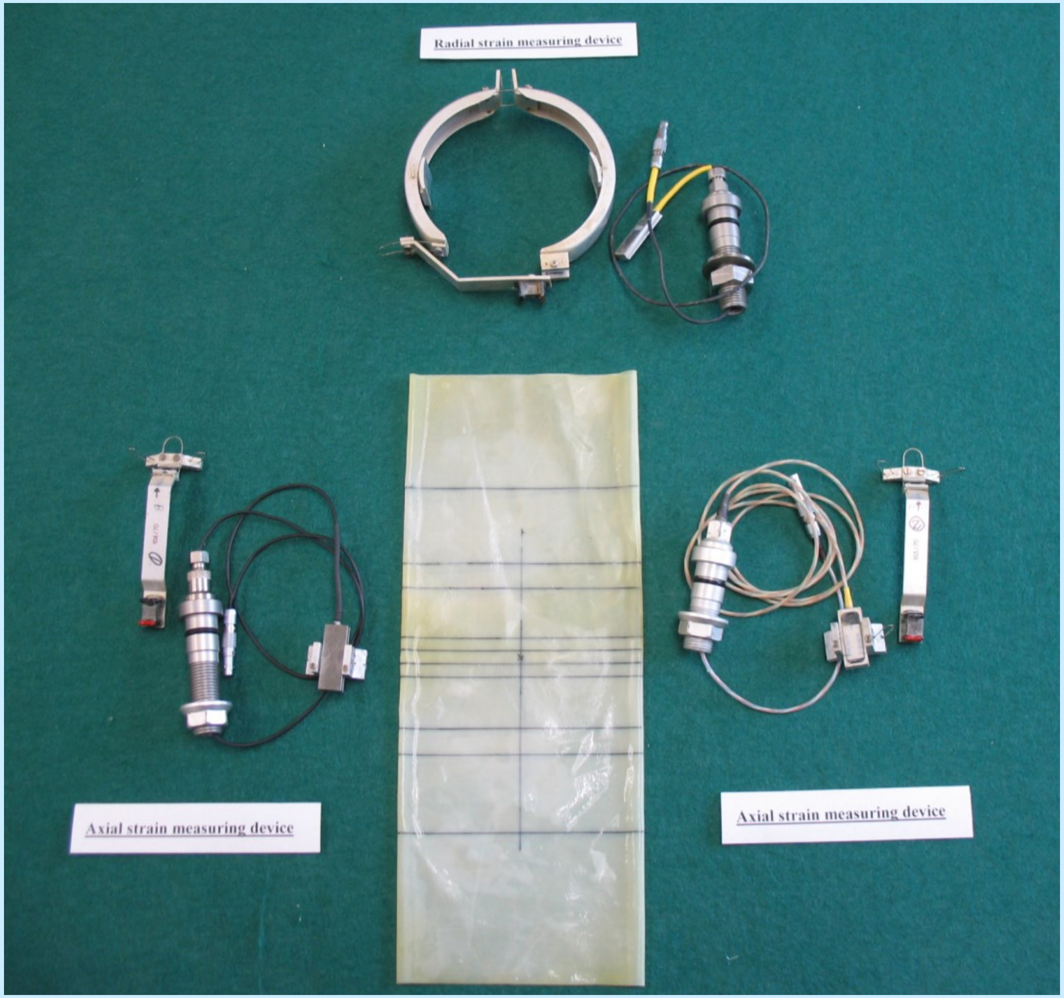
Triaxial setup

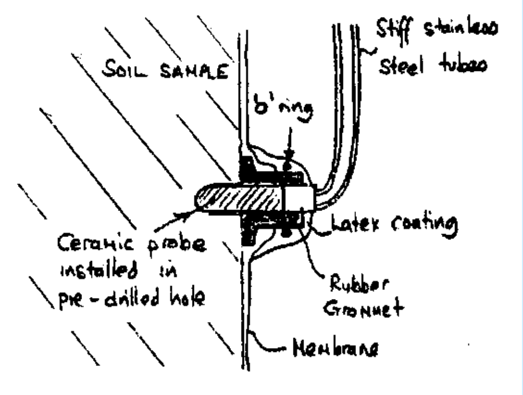
1. Force ⇨ axial stress (σ_1)
2. Axial deformation ⇨ axial strain (ε_a)
3. Pressure ⇨ cell (σ_1), back, pore (u)
4. Volume

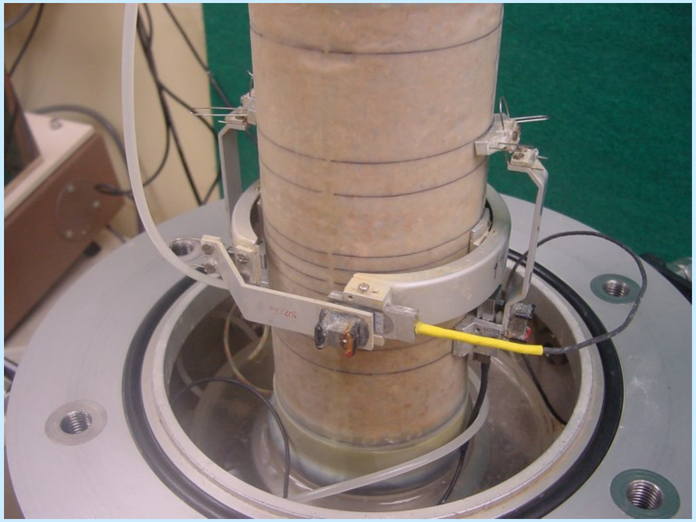
1. Specimen preparation
2. Saturation
3. Consolidation
4. Compression (shearing)

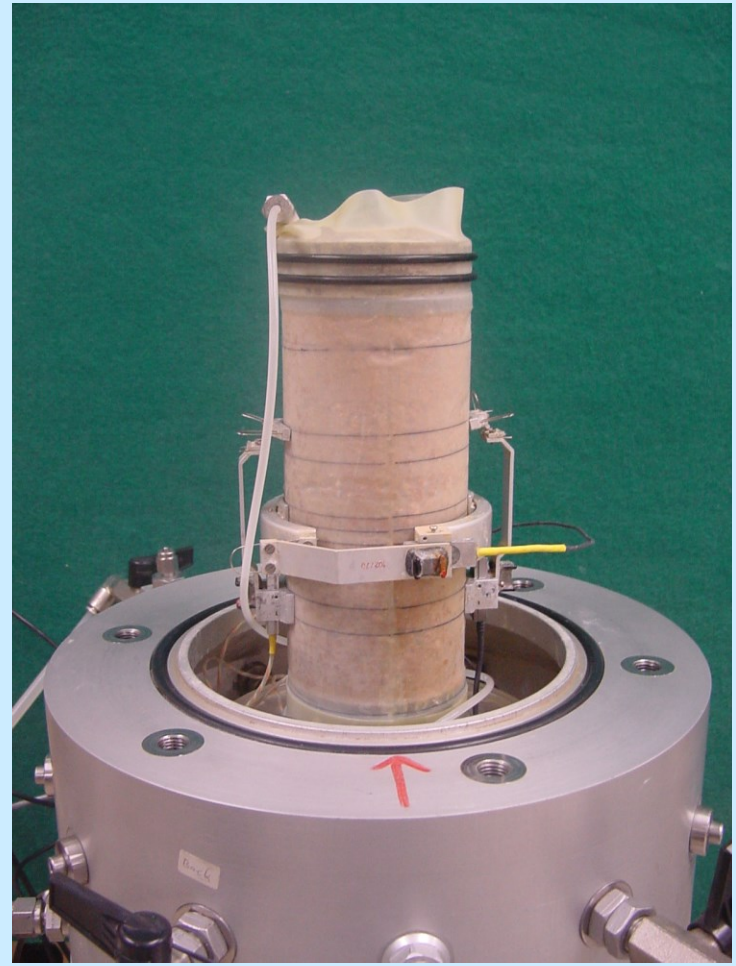


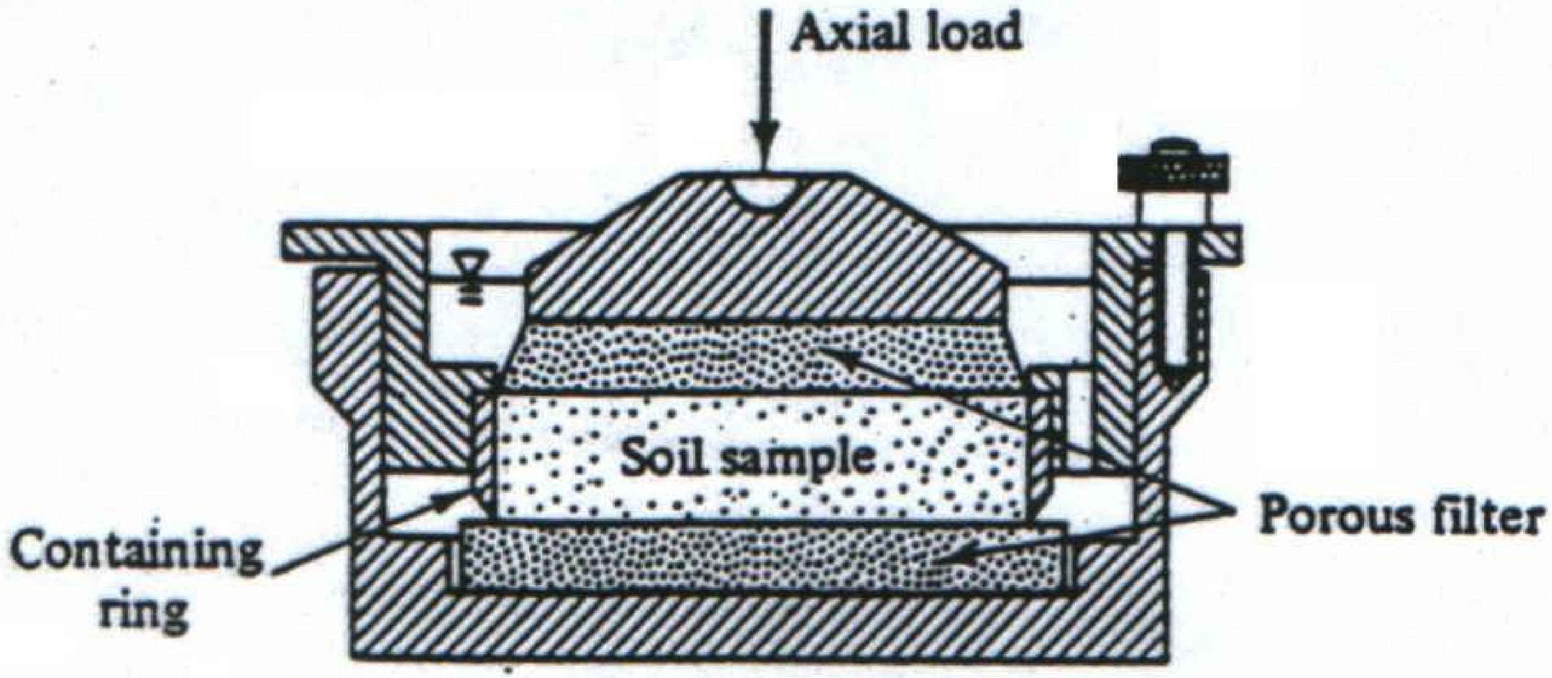
Local Axial, Radial and Pore Pressure Transducers



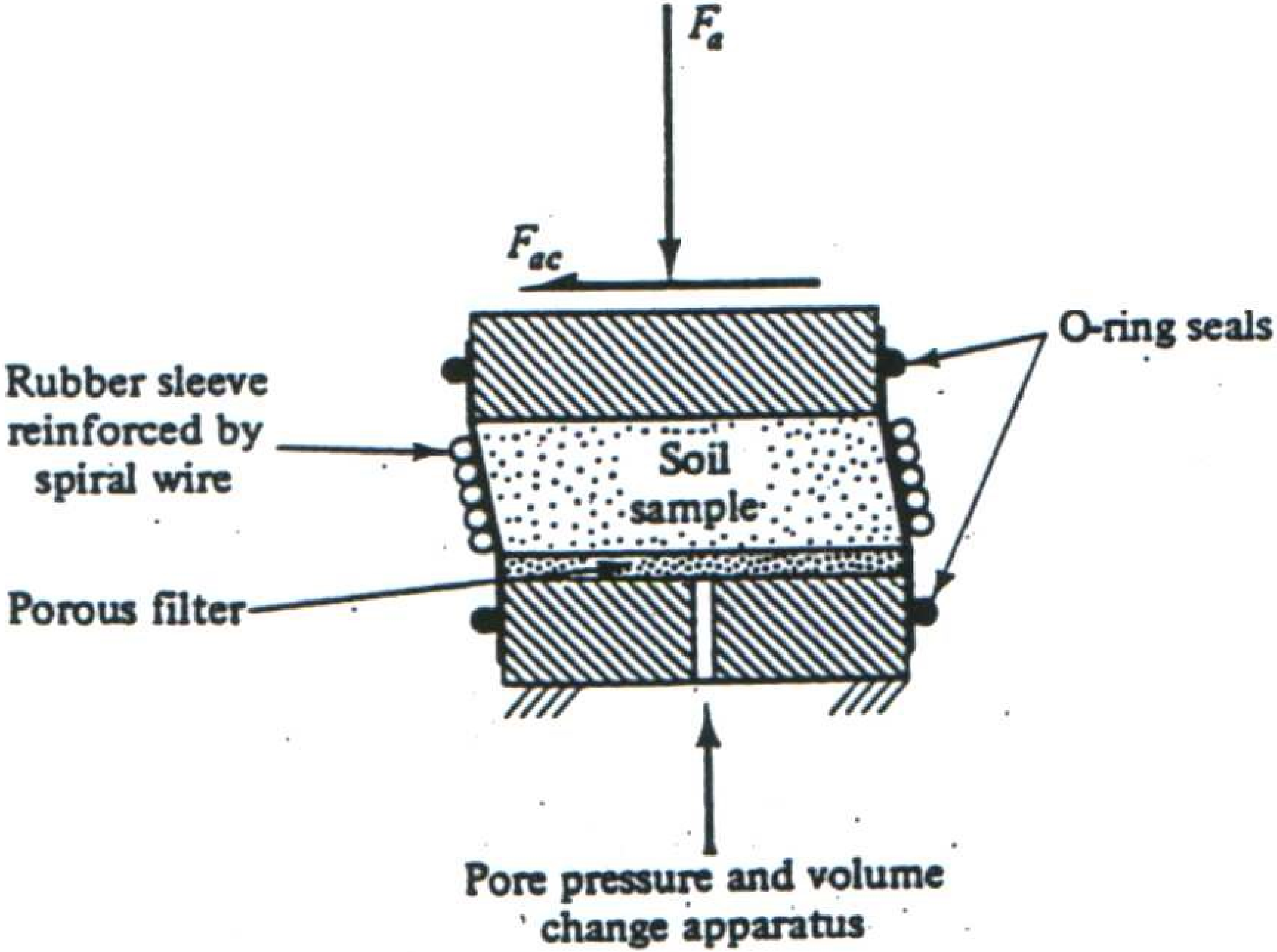








The oedometer (after BS 1377 : 1975)



NGI simple shear apparatus

In terms of the stress invariants the parameters are

$$q = (\sigma_1' - \sigma_3')$$

$\frac{1}{3}($

and

$$p = 1/3(\sigma_1' + 2\sigma_3')$$

For Odeometer type of consolidation

$$\eta = 3\left[\frac{1 - K_0}{1 + 2K_0}\right]$$

For other type of consolidation

$$\eta = 3\left[\frac{1-K}{1+2K}\right]$$

where K = $\frac{\sigma_3'}{\sigma_1'}$

Also during consolidation the stress paths are radial in nature passing through the origin in the

(q, p), (σ_1', σ_3') plots.

Thus

$$\frac{q}{p} = \frac{dq}{dp}$$

and

$$\frac{\sigma_1'}{\sigma_3'} = \frac{d\sigma_1'}{d\sigma_3'}$$

The shearing process can best be described as a process in which the soils are subjected to stress paths, which seek failure both in compression and in extension and for these cases

$$|d\eta| > 0 \qquad \textbf{where} \qquad \eta = \frac{q}{p}$$

In terms of total stresses

$$t = \frac{1}{2}(\sigma_1 - \sigma_3)$$

$$s = \frac{1}{2}(\sigma_1 + \sigma_3)$$

In terms of effective stresses

$$t' = \frac{1}{2}(\sigma_1' - \sigma_3')$$

$$s' = \frac{1}{2}\left(\sigma_1' + \sigma_3'\right)$$

$$t = t' \qquad s' = s - u$$

With respect to the Mohr- circle stress parameters total and effective stress paths can be plotted as desired. In addition to (q,p), (t' , s') plots some authors also prefer the

$$(\sigma_1{}', \sqrt{2}\sigma_3{}')$$

For axi-symmetric conditions

$$q' = (\sigma_1' - \sigma_3'),$$

$$p' = \tfrac{1}{3}(\sigma_1' + 2\sigma_3'),$$

$$\varepsilon_s = \tfrac{2}{3}(\varepsilon_1 - \varepsilon_3),$$

$$\varepsilon_v = (\varepsilon_1 + 2\varepsilon_3).$$

For plane strain conditions

$$t' = \tfrac{1}{2}(\sigma_1' - \sigma_3'),$$

$$s' = \tfrac{1}{2}(\sigma_1' + \sigma_3'),$$

$$\varepsilon_\gamma = (\varepsilon_1 - \varepsilon_3),$$

$$\varepsilon_v = (\varepsilon_1 + \varepsilon_3).$$

Strain Increment

The incremental axial strain is given by

$$\dot{\varepsilon}_1 = \frac{\dot{l}}{l}$$

$\dot{l}$ **is the increment in height and l is the current height.**

When integrated we have

$$\varepsilon_1 = \ln \frac{l_0}{l}$$

l_0 is the initial length and l is the current length.

Similarly the incremental radial strain is given by

$$\dot{\varepsilon}_3 = \frac{\dot{r}}{r}$$

$\dot{r}$ **is increment in the radius and r is the current radius.**

When integrated we have

$$\varepsilon_3 = \ln \frac{r_0}{r}$$

r_0 is the initial radius and r is the current radius.

Similarly the incremental volumetric strain $\dot{\varepsilon}_V$ is given by

$$\dot{\varepsilon}_V = \frac{\dot{V}}{V}$$

$\dot{V}$ **is the increment in volume and V is the current volume.**

The incremental volumetric strain is also same as

$$\frac{\dot{e}}{(1+e_0)}$$

$\dot{e}$ **is the increment in voids ratio and** e_0 **is the current voids ratio.**

The volumetric strain ε_v can be integrated as

$$\varepsilon_v = \ln\left(\frac{V_0}{V}\right) = \ln\left(\frac{(1+e_0)}{(1+e)}\right)$$

The incremental volumetric and shear strains are related to the incremental axial and radial strains As in the next slide

$$\dot{\varepsilon}_v = \dot{\varepsilon}_1 + 2\dot{\varepsilon}_3$$

$$\dot{\varepsilon}_s = \frac{2}{3}\left(\dot{\varepsilon}_1 - \dot{\varepsilon}_3\right)$$

Work Equation

$$\sigma_1{}'\dot{\varepsilon}_1 + 2\sigma_3{}'\dot{\varepsilon}_3 = p\dot{\varepsilon}_v + q\dot{\varepsilon}_s$$

Elastic Stress Strain Relationship

For axi-symmetric conditions Atkinson & Bransby have derived the following relationships

$$\delta\varepsilon_v = \frac{1}{K'}\,\delta p' + 0.\,\delta q',$$

$$\delta\varepsilon_s = 0.\,\delta p' + \frac{1}{3G'}\,\delta q'.$$

For plane strain conditions the relationship becomes

$$\delta\epsilon_{\gamma} = \frac{2(1+\nu')}{3K'}\delta s' + 0.\delta t',$$

$$\delta\epsilon_{\varsigma} = 0.\delta s' + \frac{1}{G'}\delta t'.$$

For plane strain conditions

$$t' = \tfrac{1}{2}(\sigma'_1 - \sigma'_3),$$

$$s' = \tfrac{1}{2}(\sigma'_1 + \sigma'_3),$$

$$\varepsilon_\gamma = (\varepsilon_1 - \varepsilon_3),$$

$$\varepsilon_v = (\varepsilon_1 + \varepsilon_3).$$

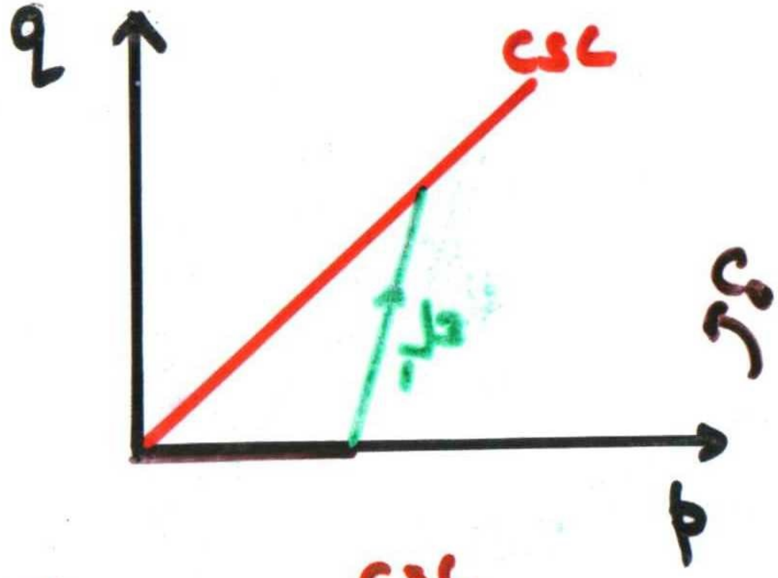
For axi-symmetric conditions

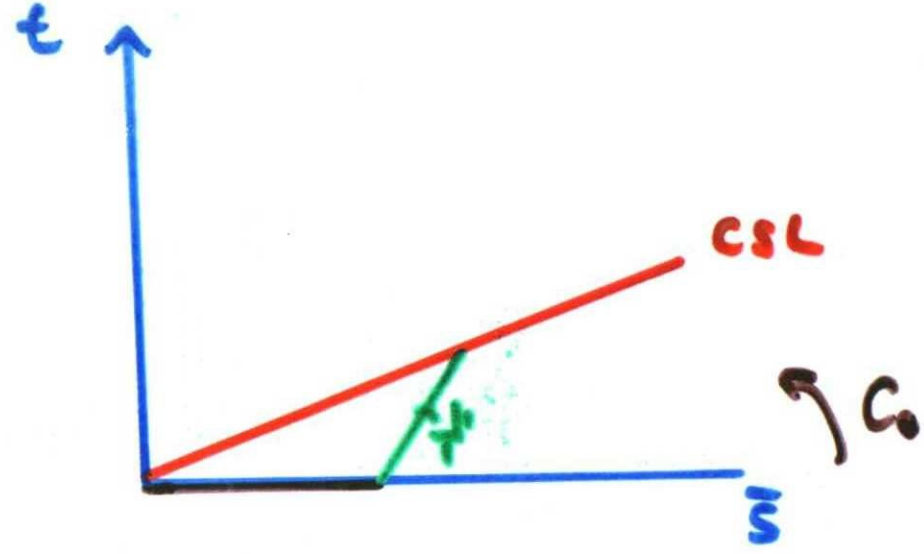
$$q' = (\sigma_1' - \sigma_3'),$$

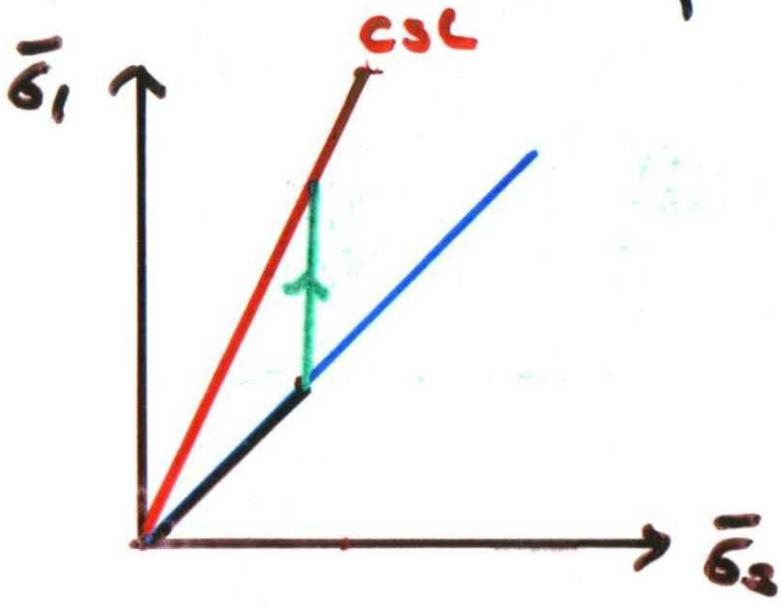
$$p' = \tfrac{1}{3}(\sigma_1' + 2\sigma_3'),$$

$$\varepsilon_s = \tfrac{2}{3}(\varepsilon_1 - \varepsilon_3),$$

$$\varepsilon_v = (\varepsilon_1 + 2\varepsilon_3).$$



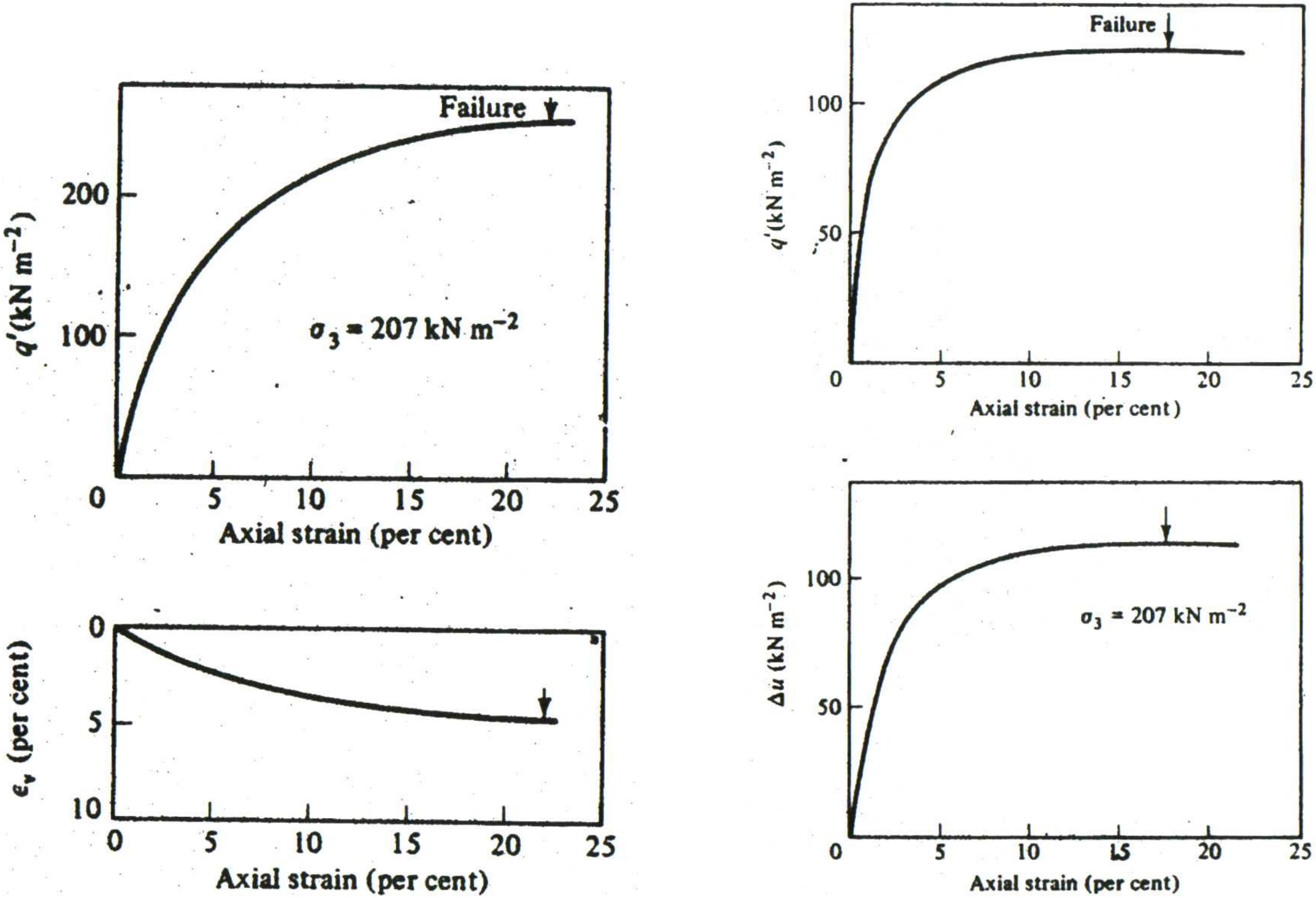




CID Test

Drained Path

Volumetric strain in drained test and excess pore pressure in undrained test (NC state)



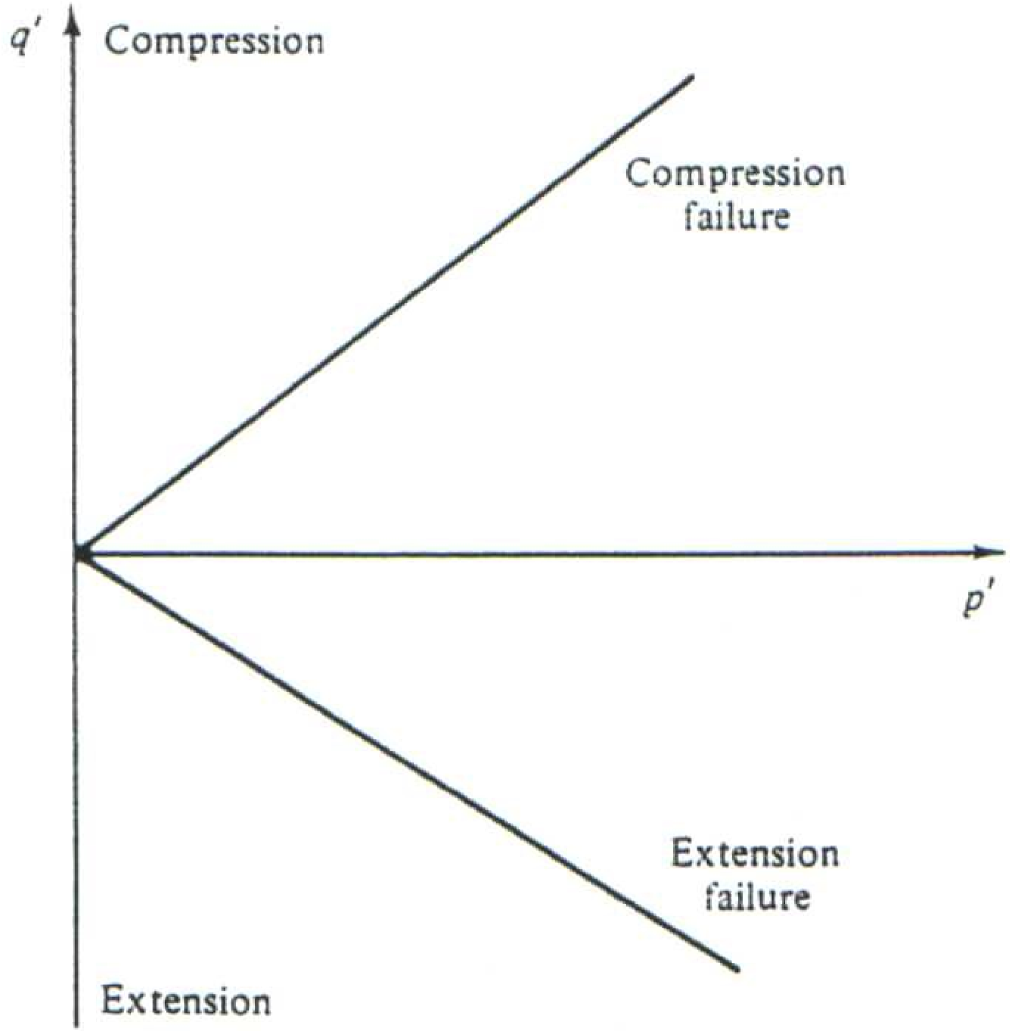
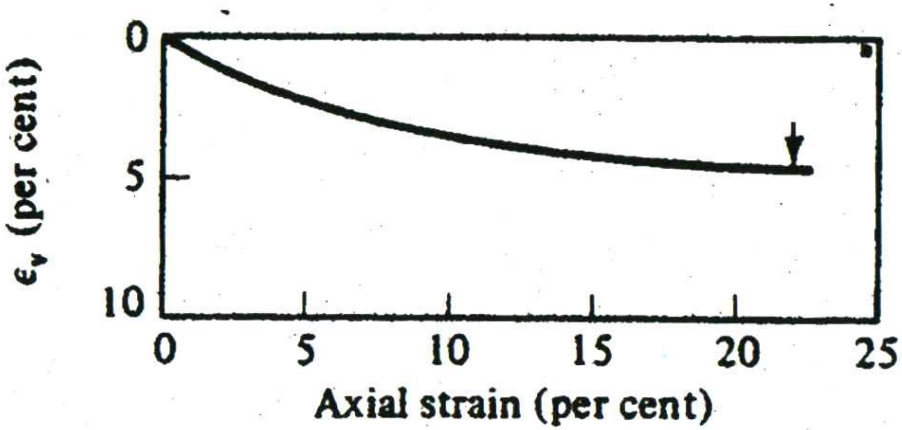
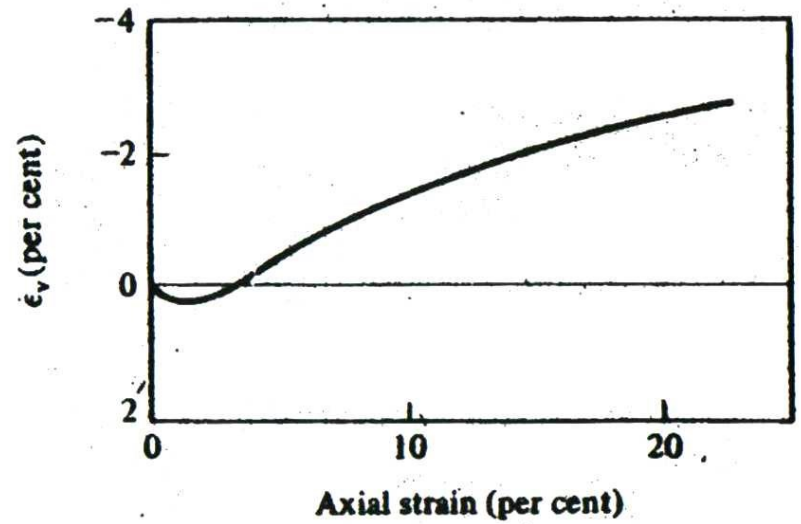


Figure 14-13 Locus of failure states for compression and extension tests

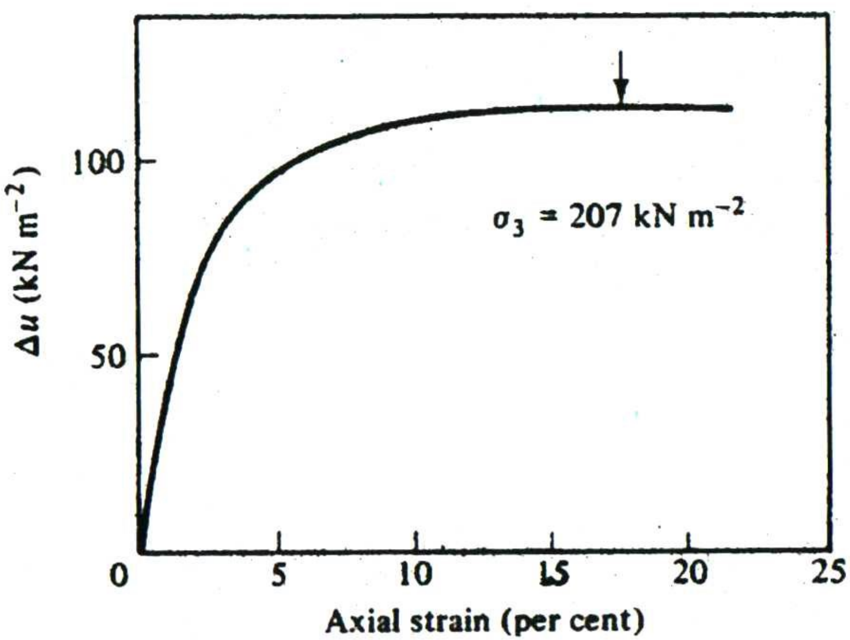
Volumetric strain & excess pore pressures

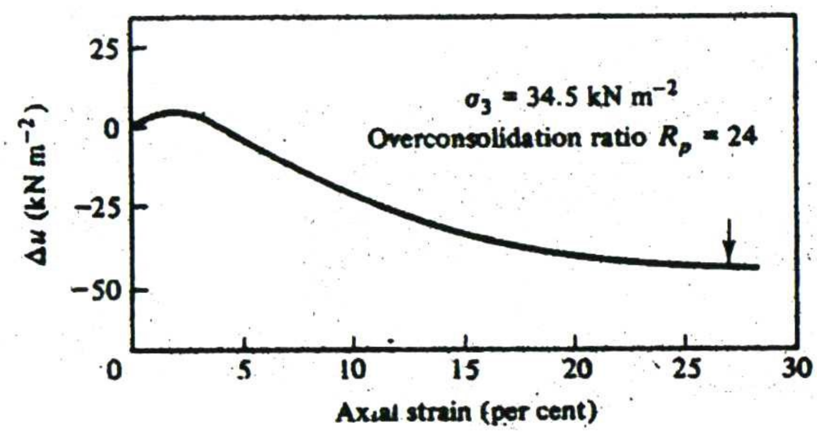


Normally consolidated clay

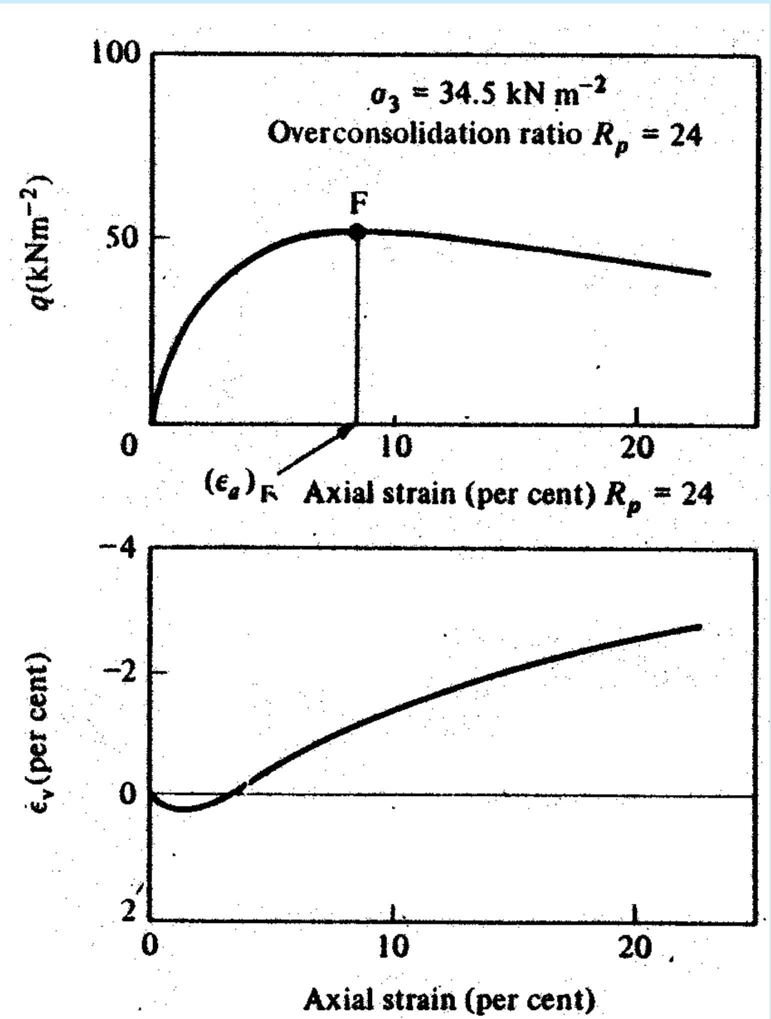


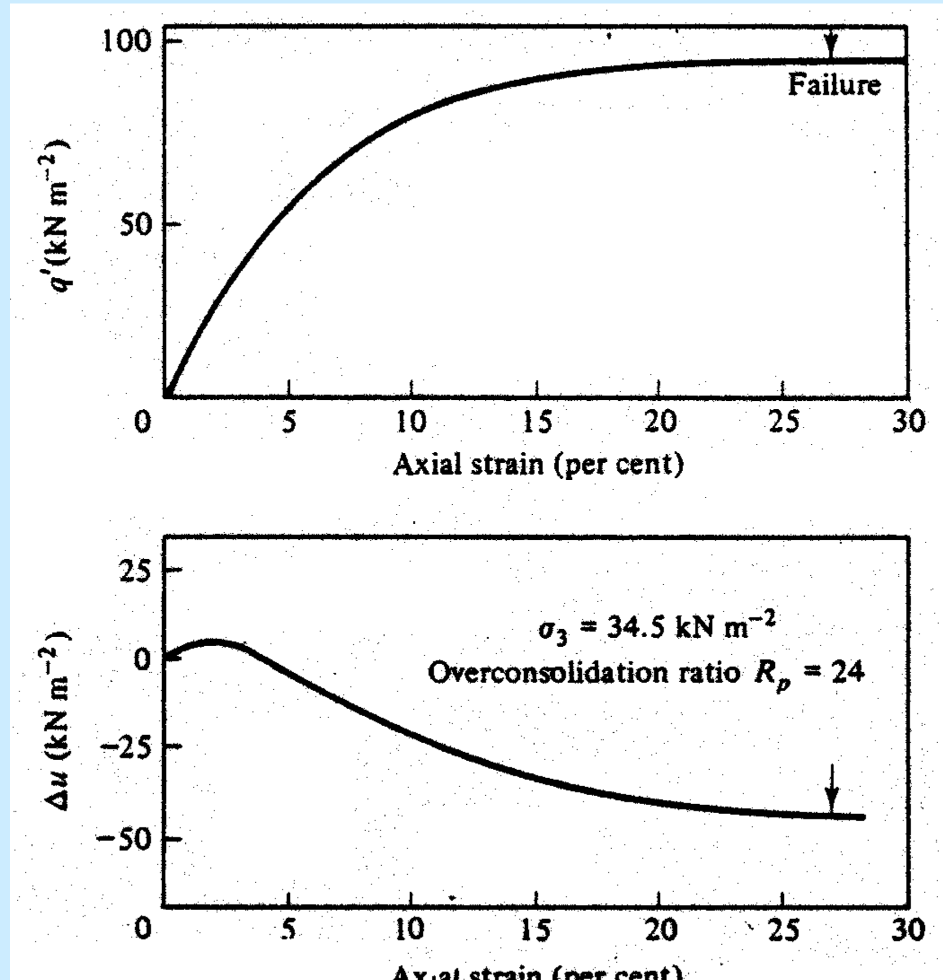
Heavily over-consolidated clay



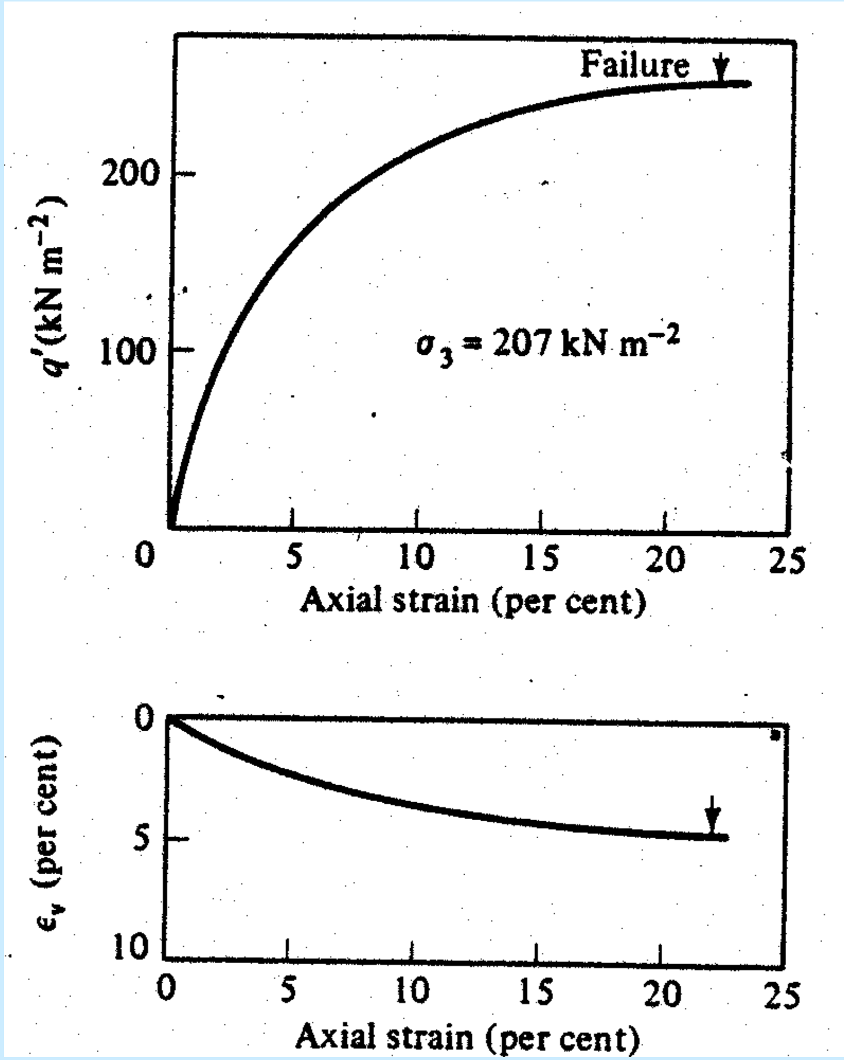


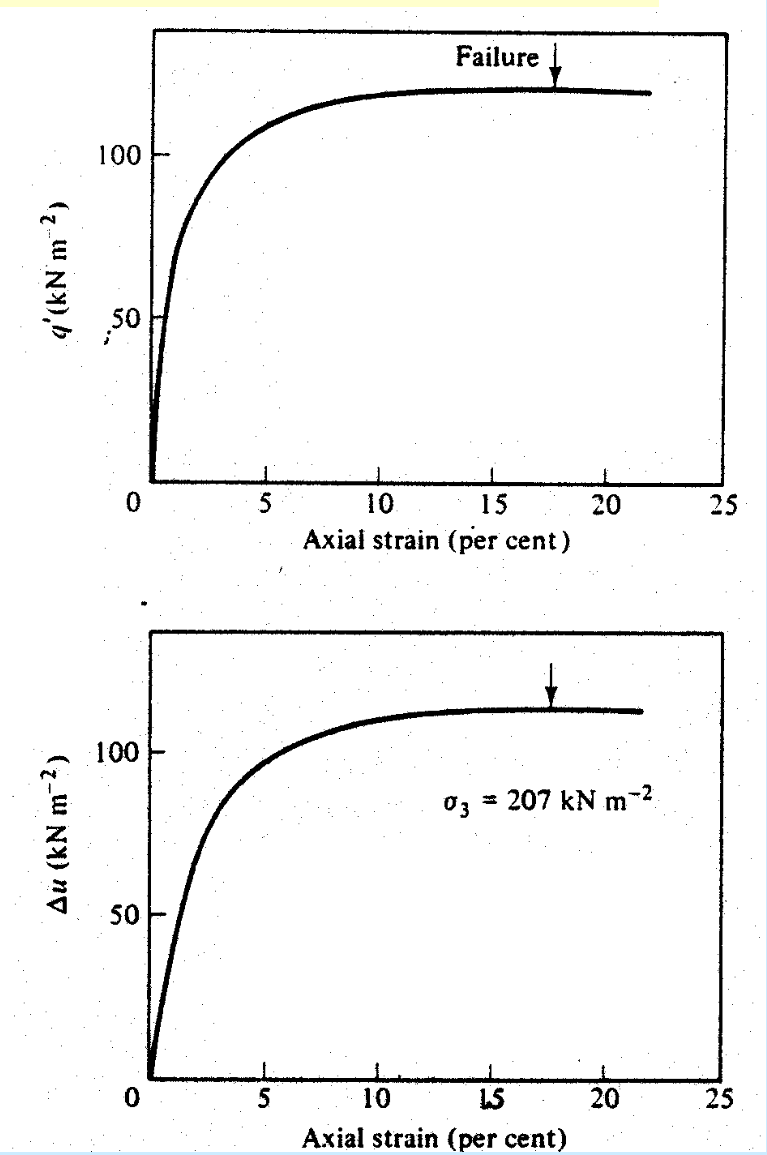
Volumetric dilation in heavily overconsolidated clays in drained case and large excess negative pore pressure in undrained case





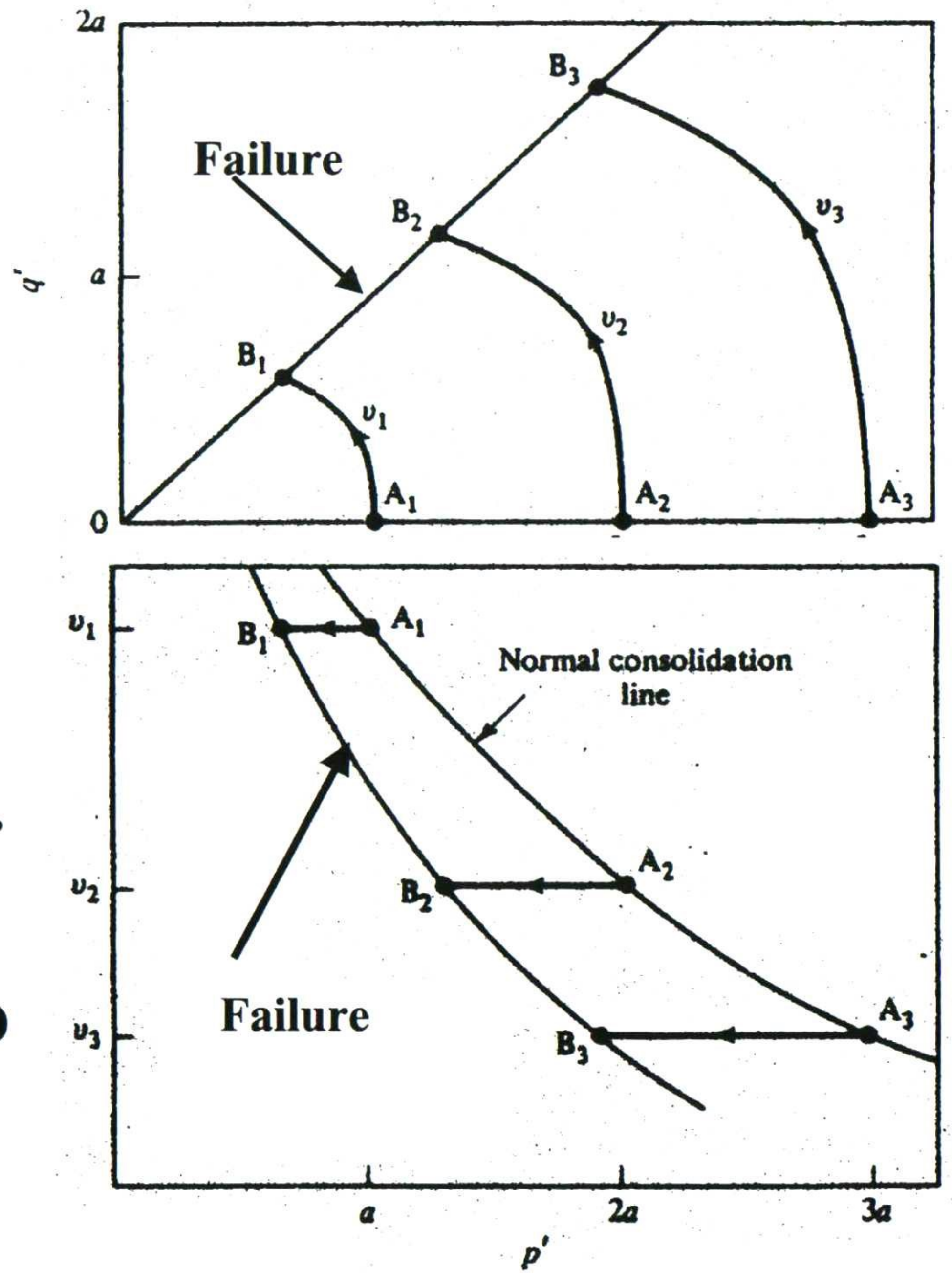
Volumetric strain in drained test and excess pore pressure in undrained test (NC state)



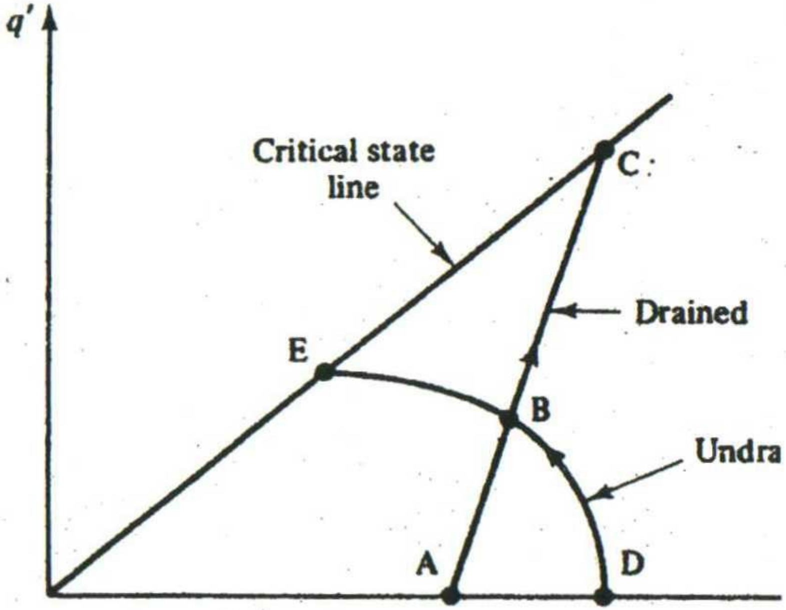


Undrained tests Roscoe, Schofield & Wroth way of Interpretation

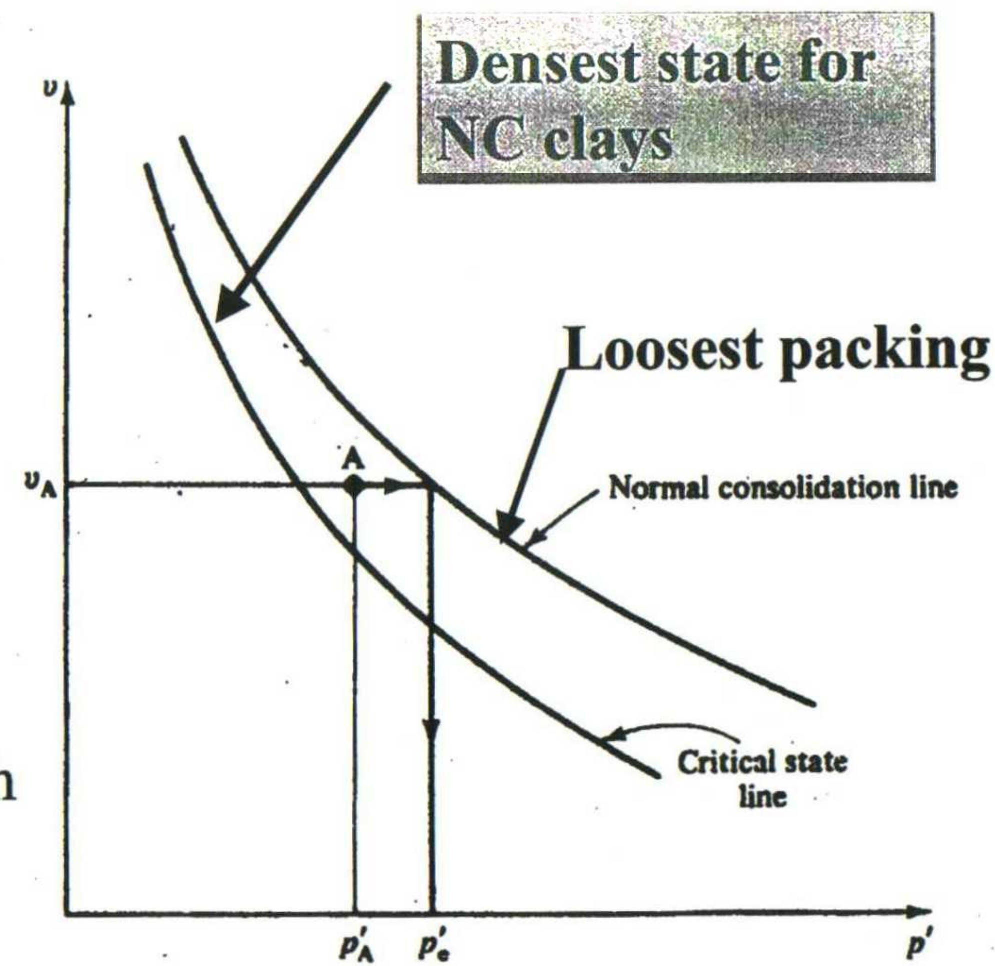
Use of (q,p) plot - stress path & Voids ratio or water content or specific volume with q and p plots

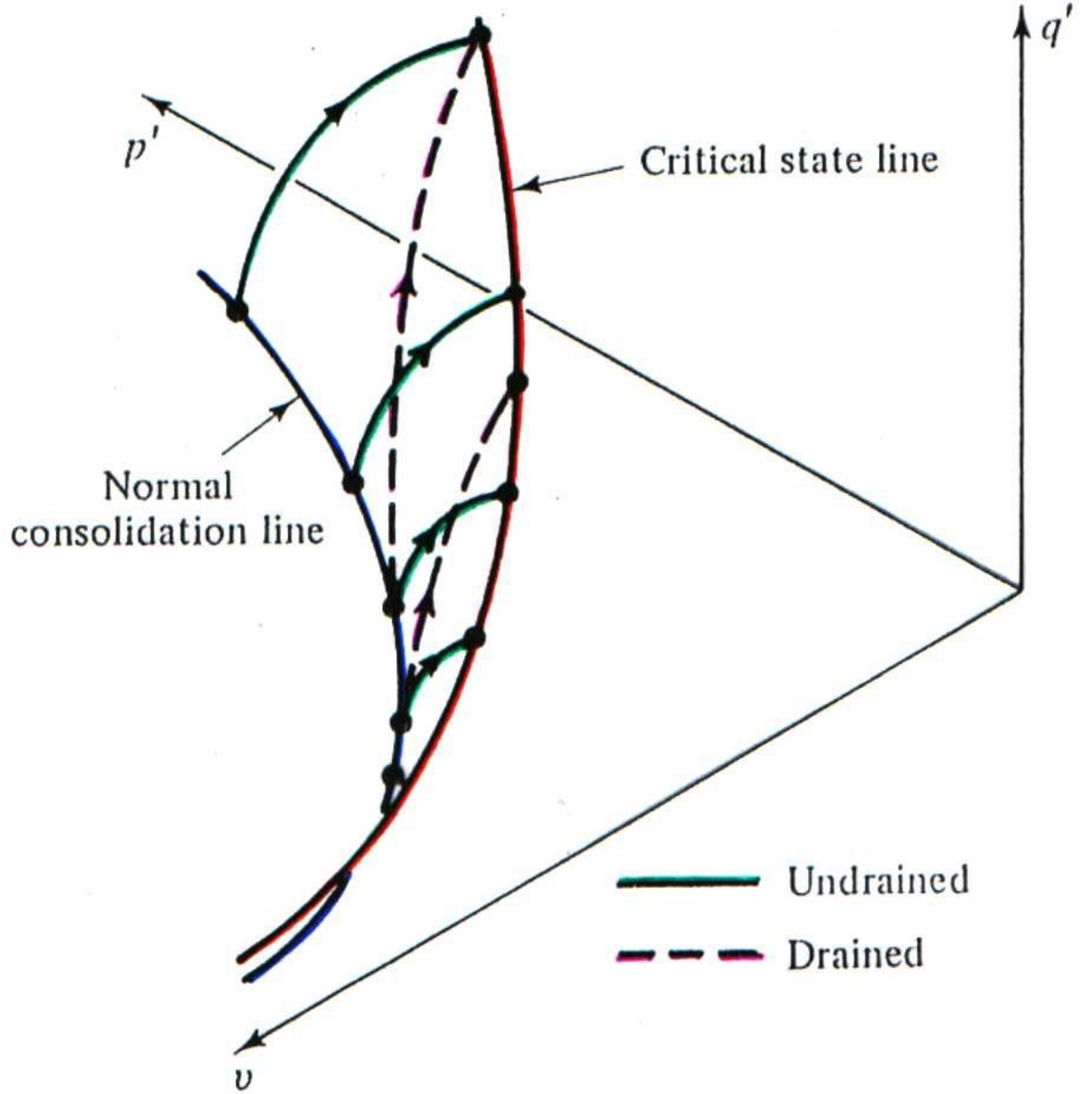


Hvorslev's mean equivalent pressure, p_e

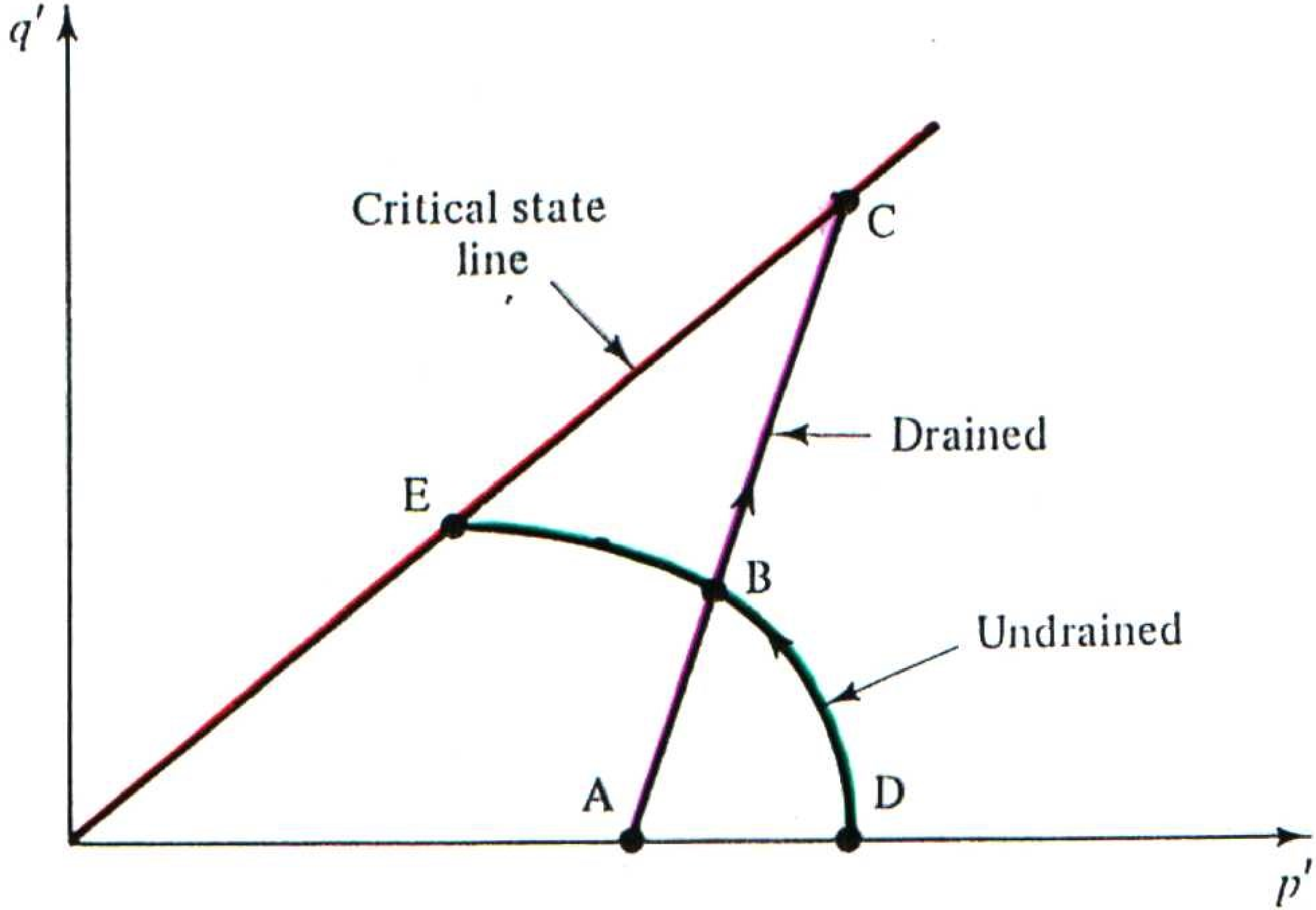


When sample A is sheared to B, at B the mean equivalent pressure is the same as the consolidation pressure corresponding to point D. DBE is an undrained stress path

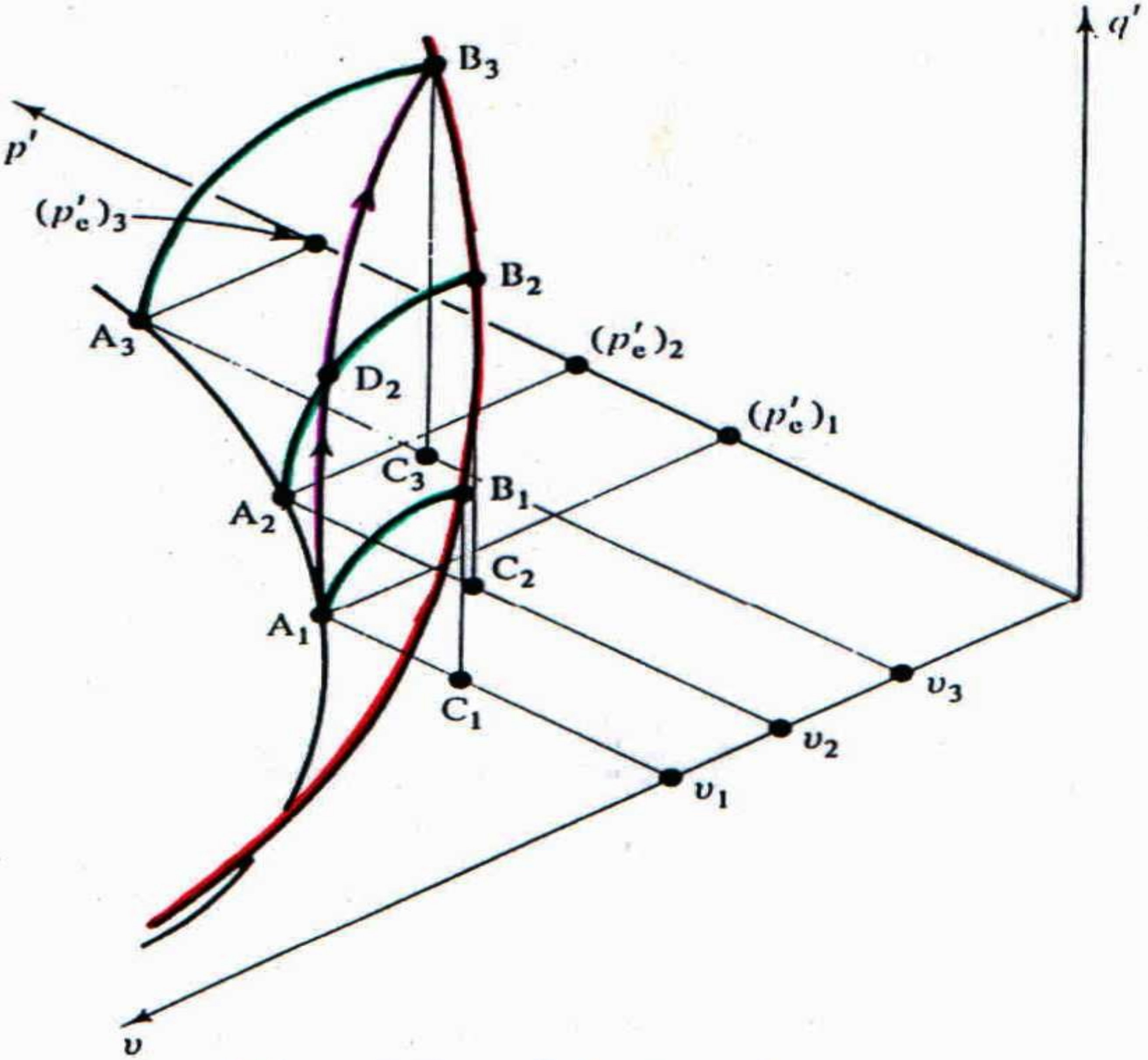




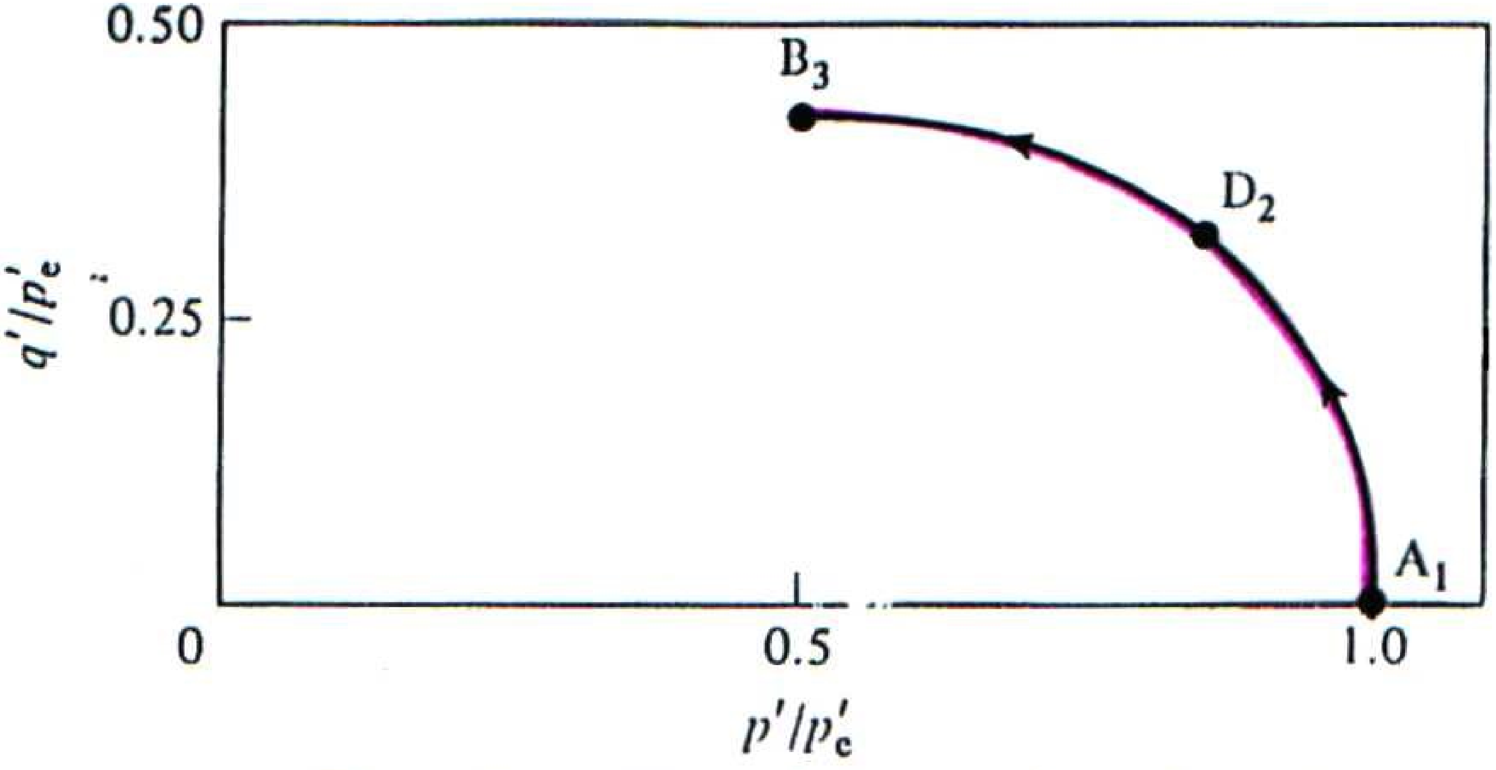
Families of drained and undrained tests in $q' : p' : v$ space



Drained and undrained paths in $q' : p'$ space



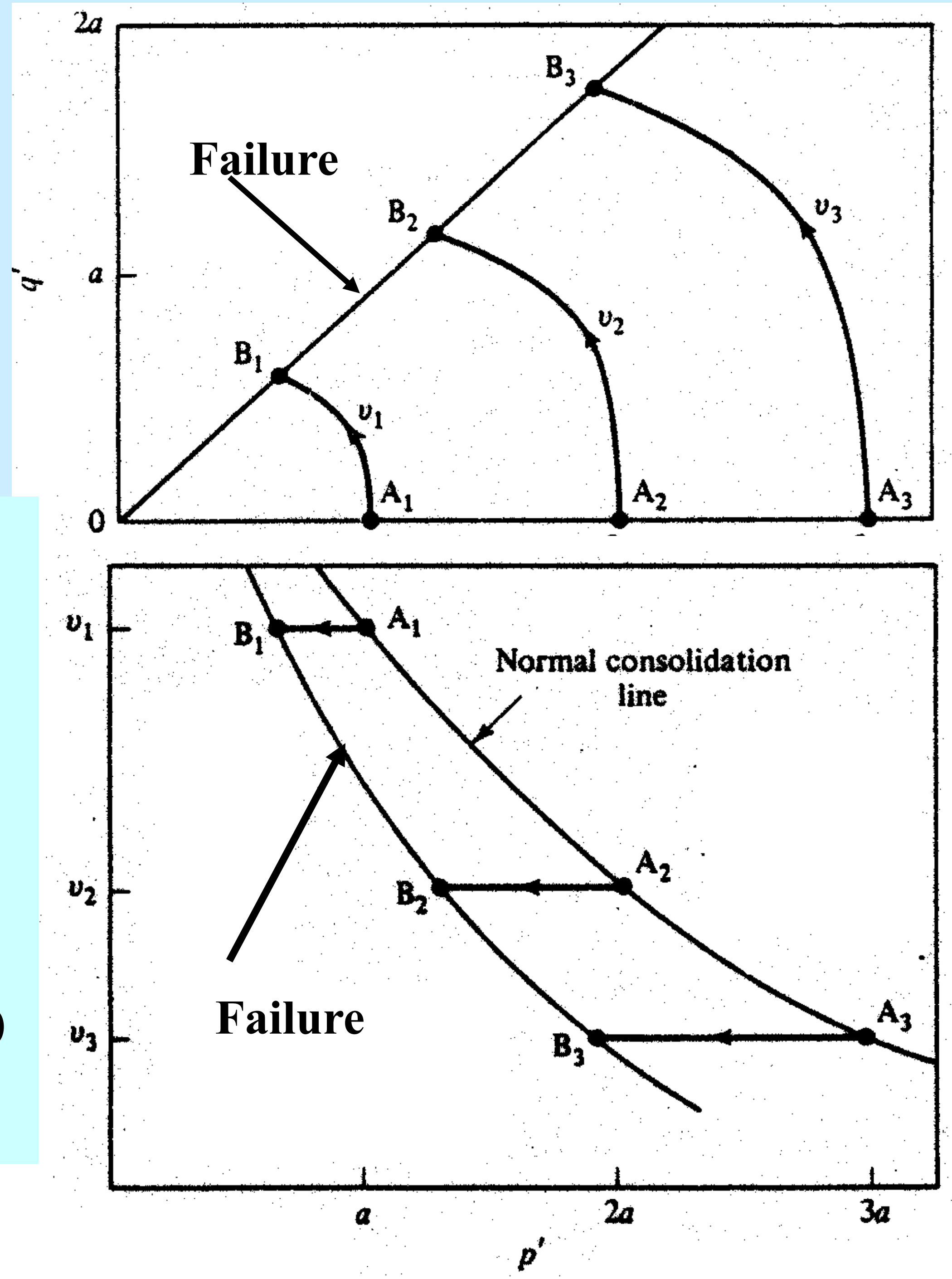
The path for a drained test in $q' : p' : v$ space



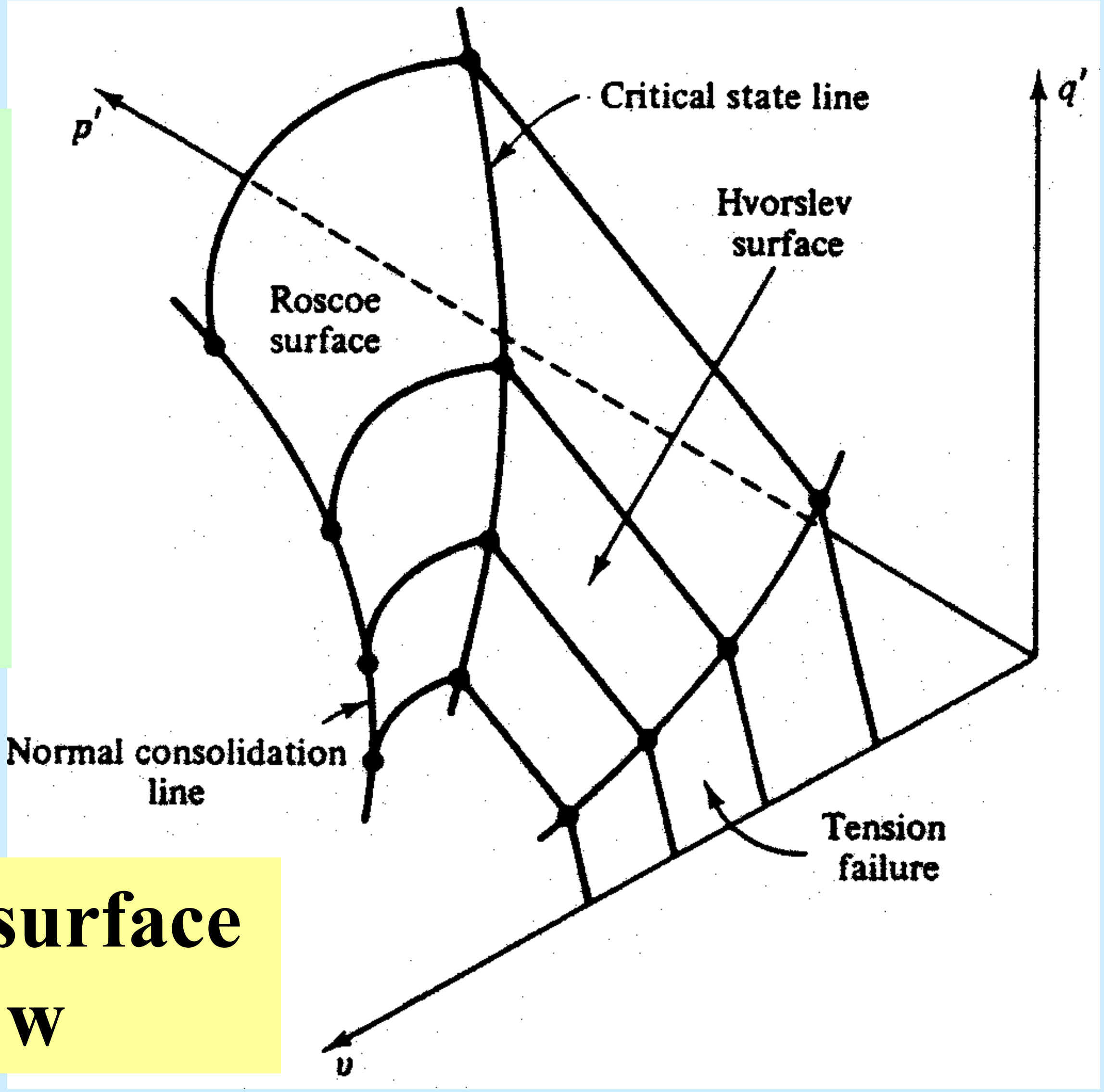
The path in $q'/p'_c : p'/p'_c$ space for a drained test

Undrained tests Roscoe, Schofield & Wroth way of Interpretation

Use of (q,p) plot - stress path & Voids ratio or water content or specific volume with q and p plots

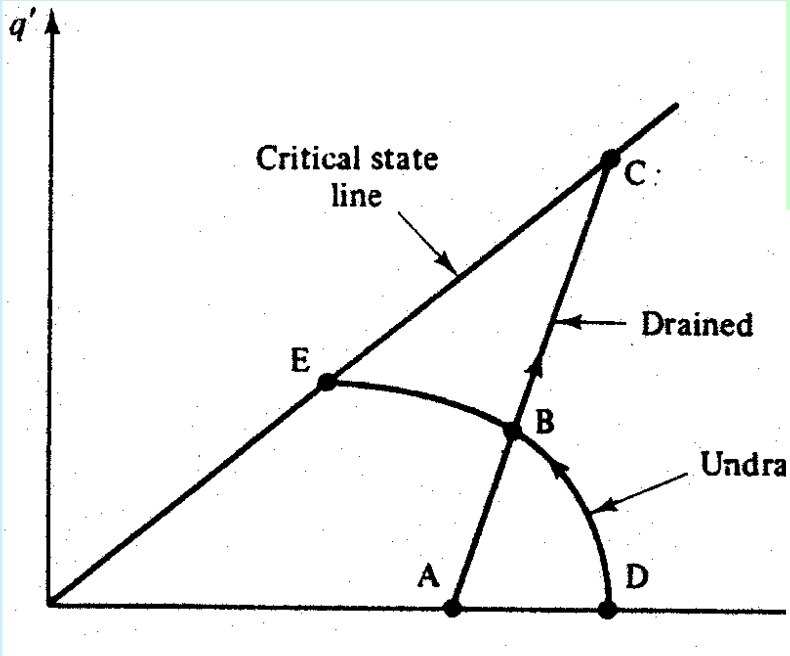


Roscoe Schofield & Wroth contribution

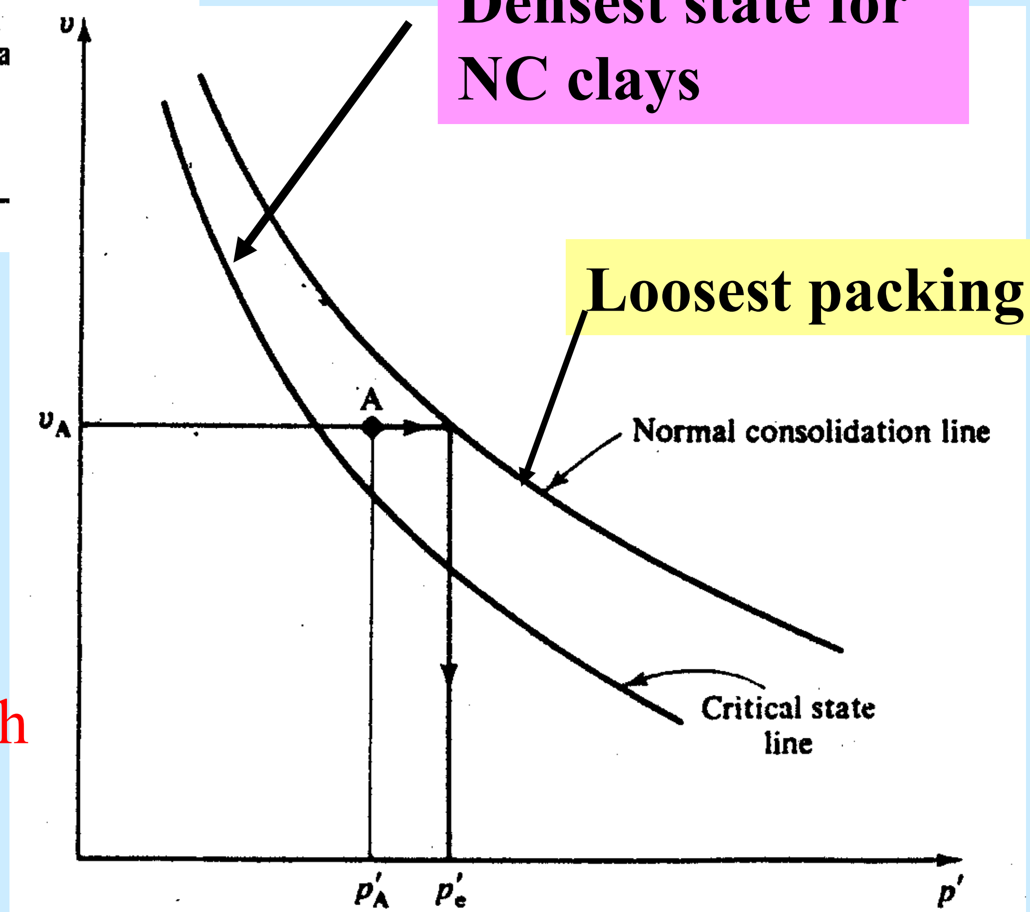


State boundary surface in (q,p, e or v or w

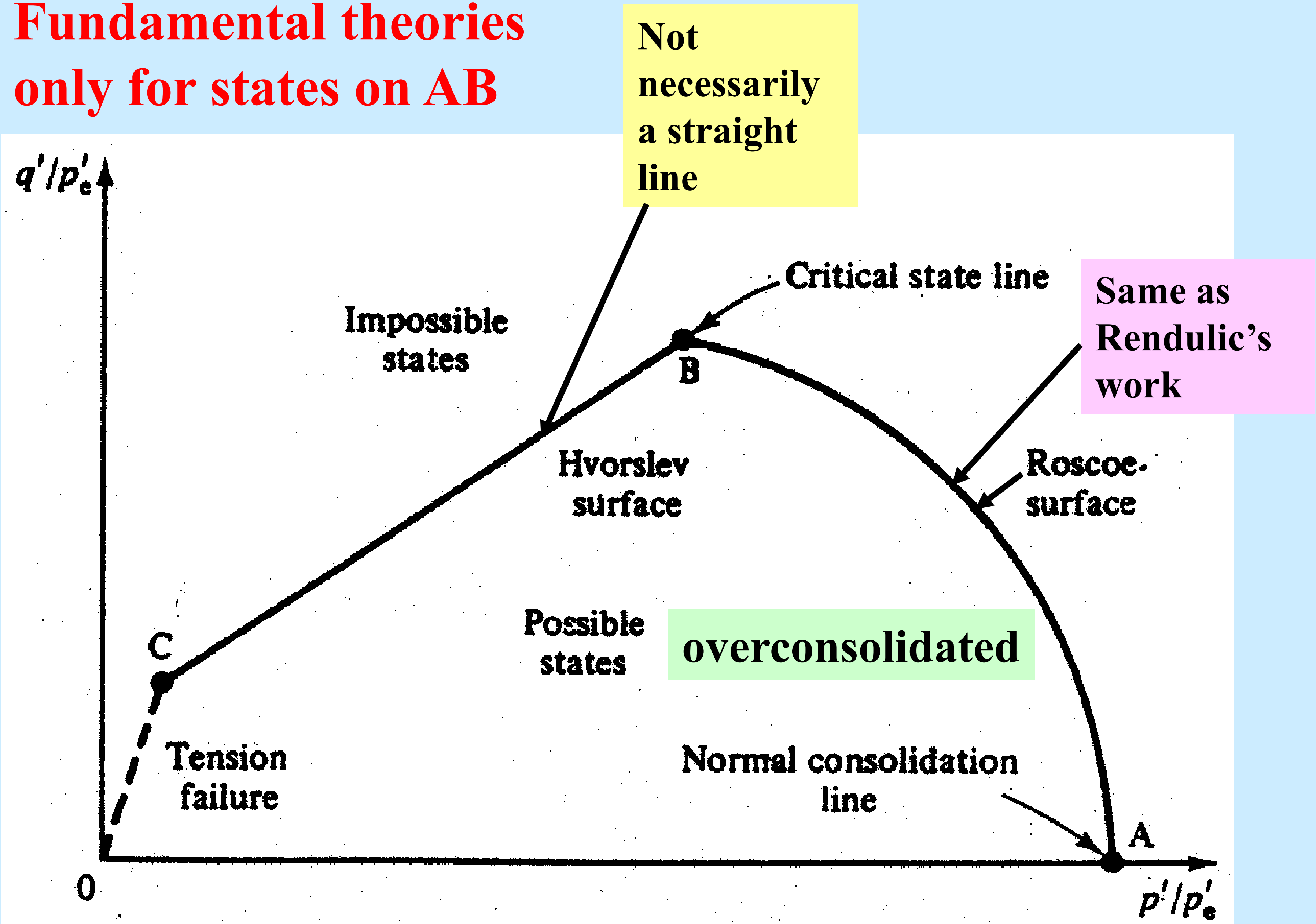
Hvorslev's mean equivalent pressure, p_e



When sample A is sheared to B, at B the mean equivalent pressure is the same as the consolidation pressure corresponding to point D.
DBE is an undrained stress path



Fundamental theories only for states on AB



State Surface in 2-D

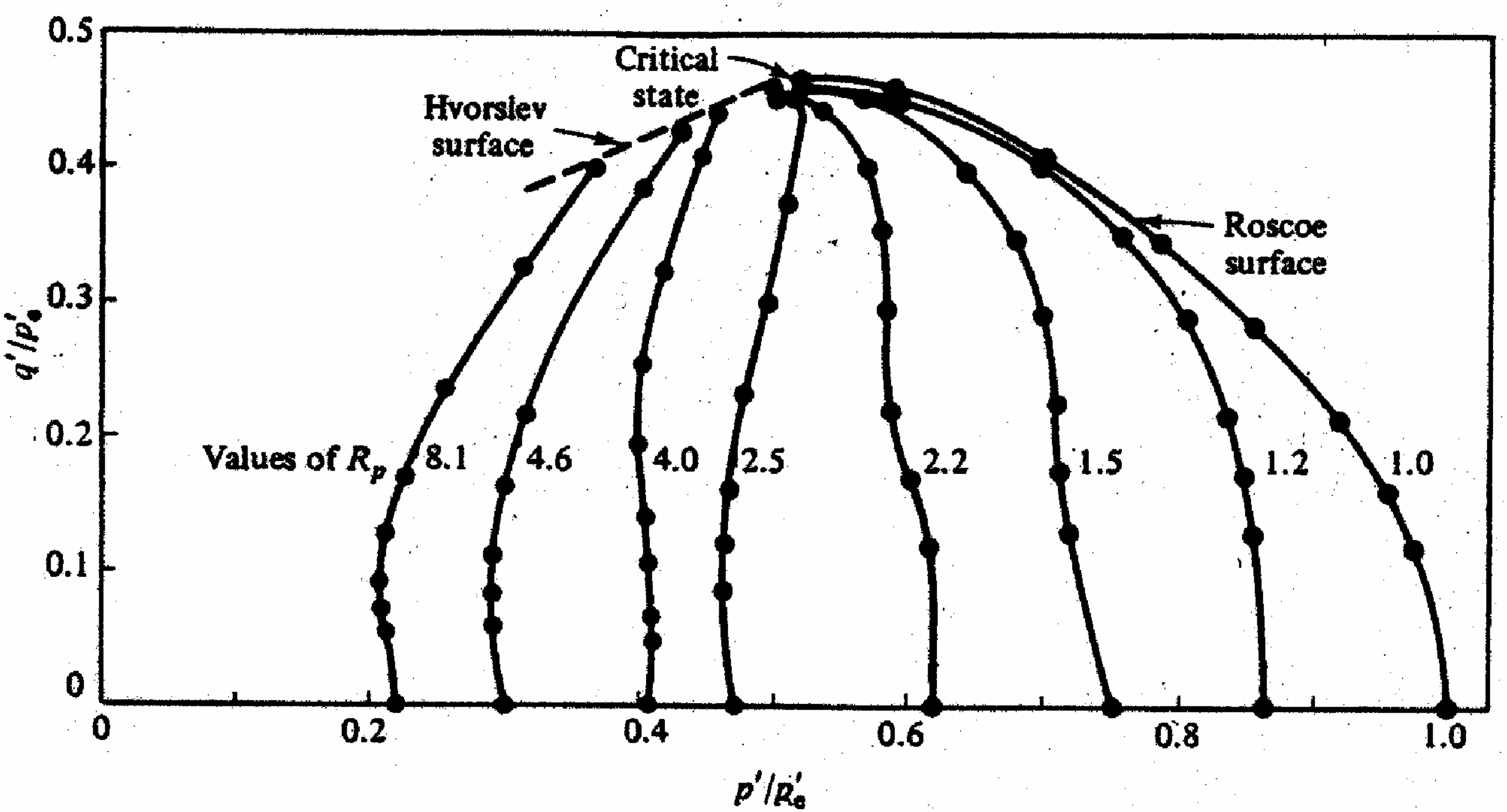
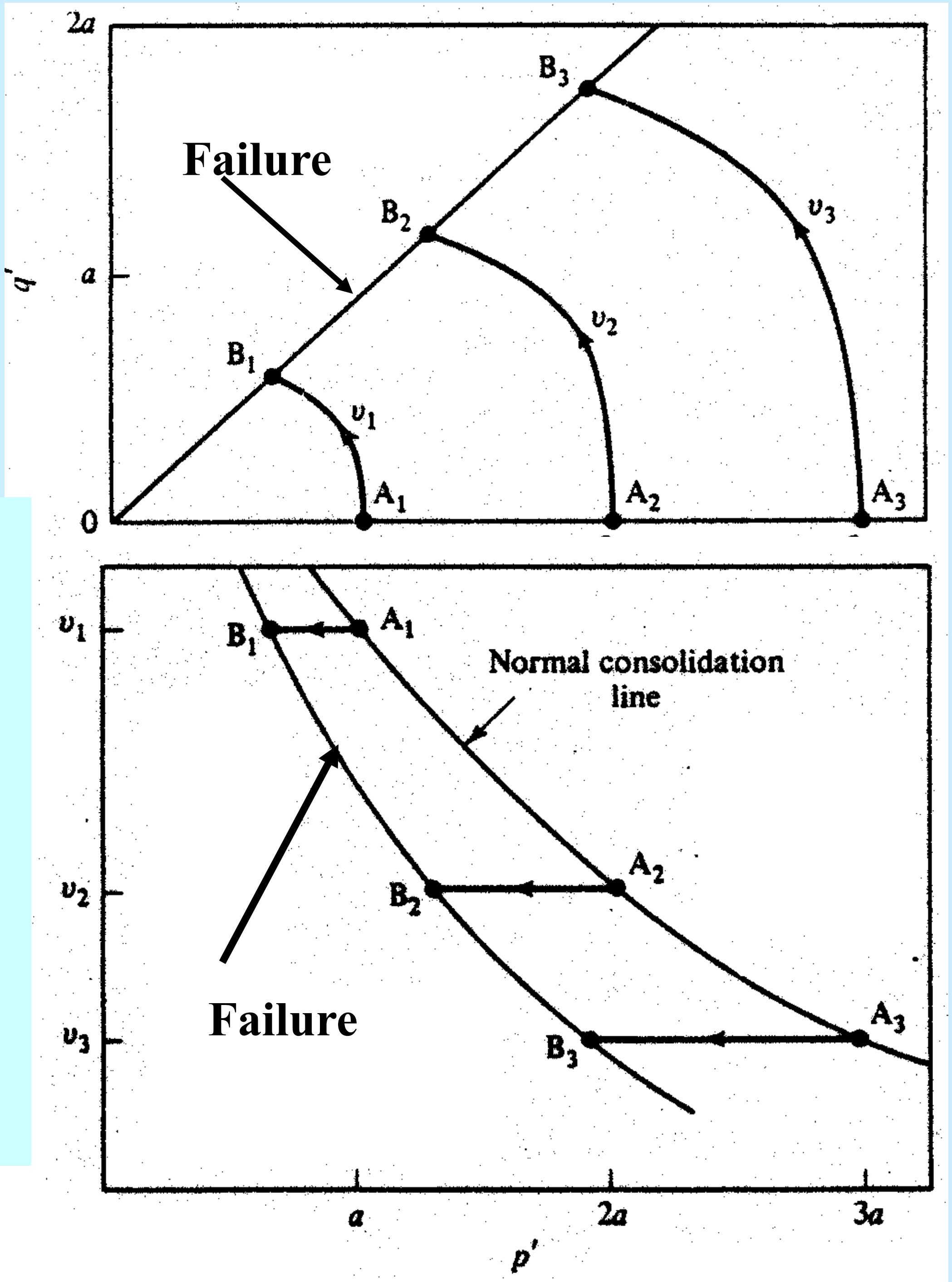


Figure 2.18: Undrained stresspaths in overconsolidated clays

Undrained tests Roscoe, Schofield & Wroth way of Interpretation

Use of
(q,p) plot - stress path &
Voids ratio or water content or specific volume with q and p
 plots



Stress-strain behaviour

Undrained behaviour

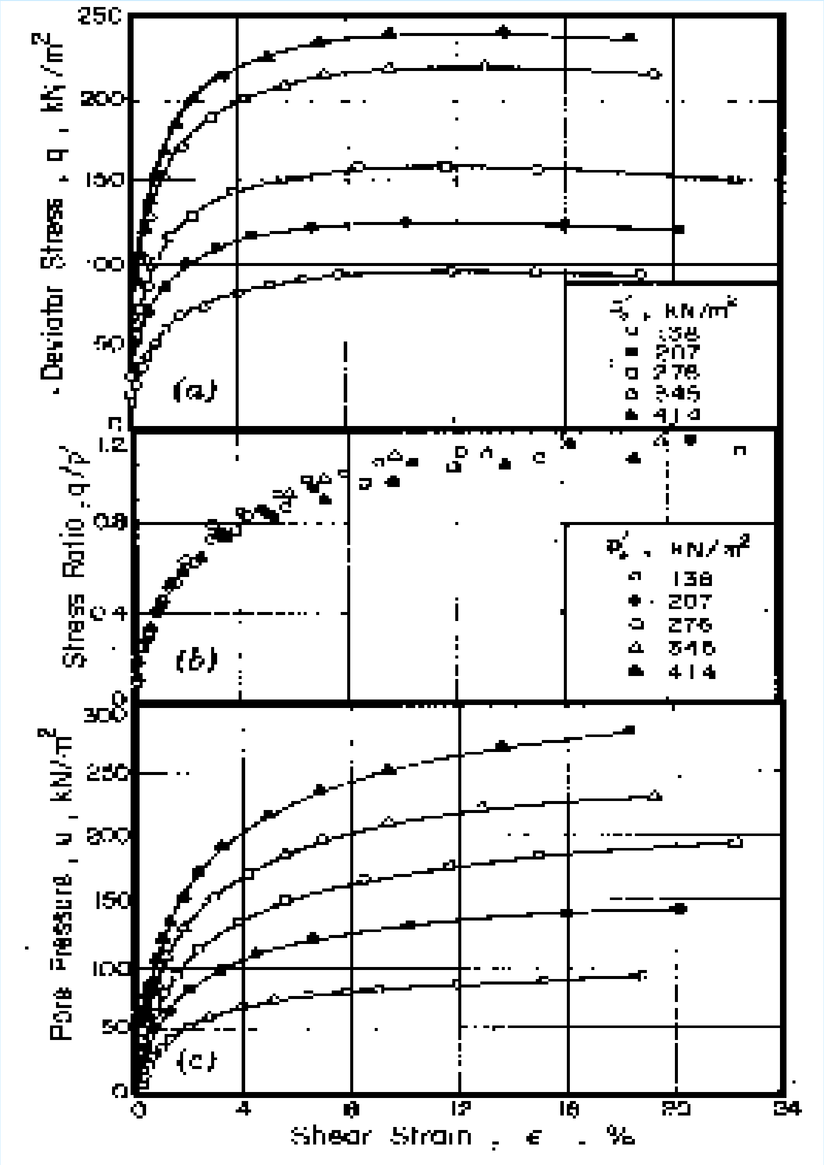


Figure 3.7 Pore pressure, shear strain and deviator stress stress ratio shear strain relationships in undrained shear

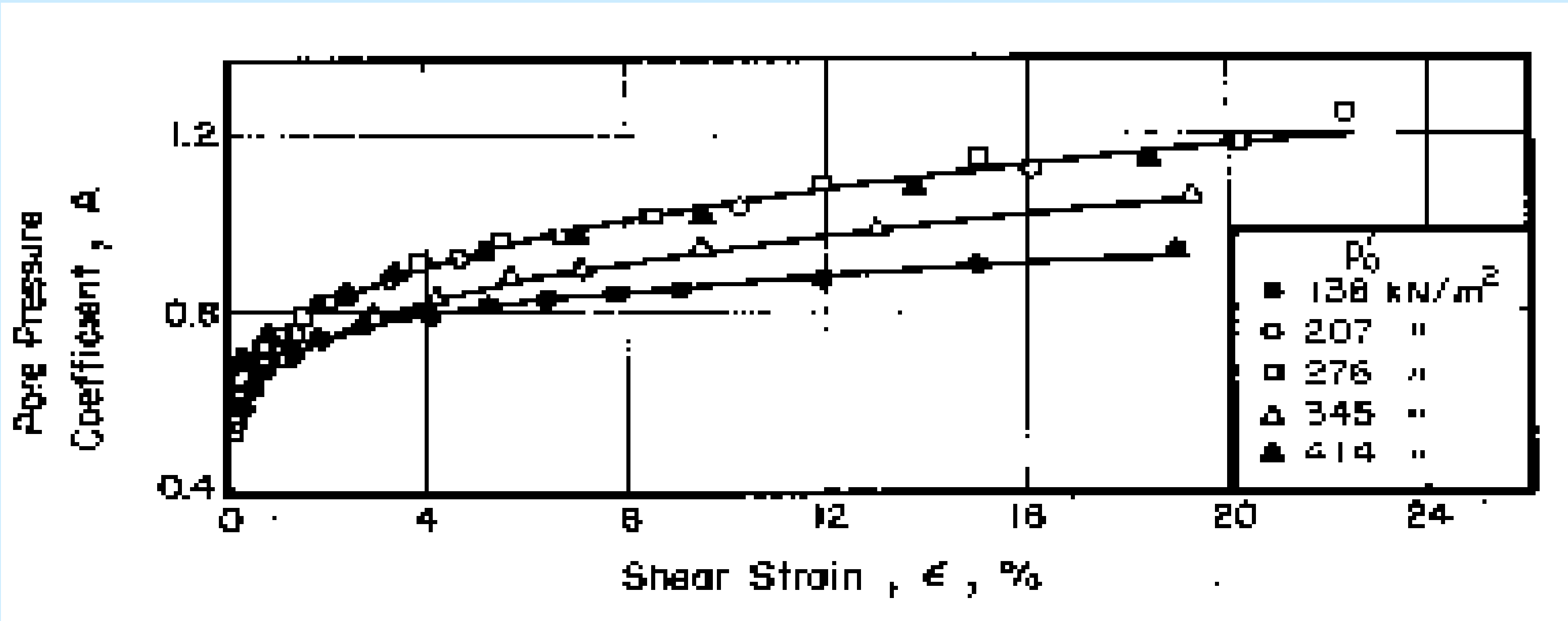


Figure 3.8: Pore pressure parameter A

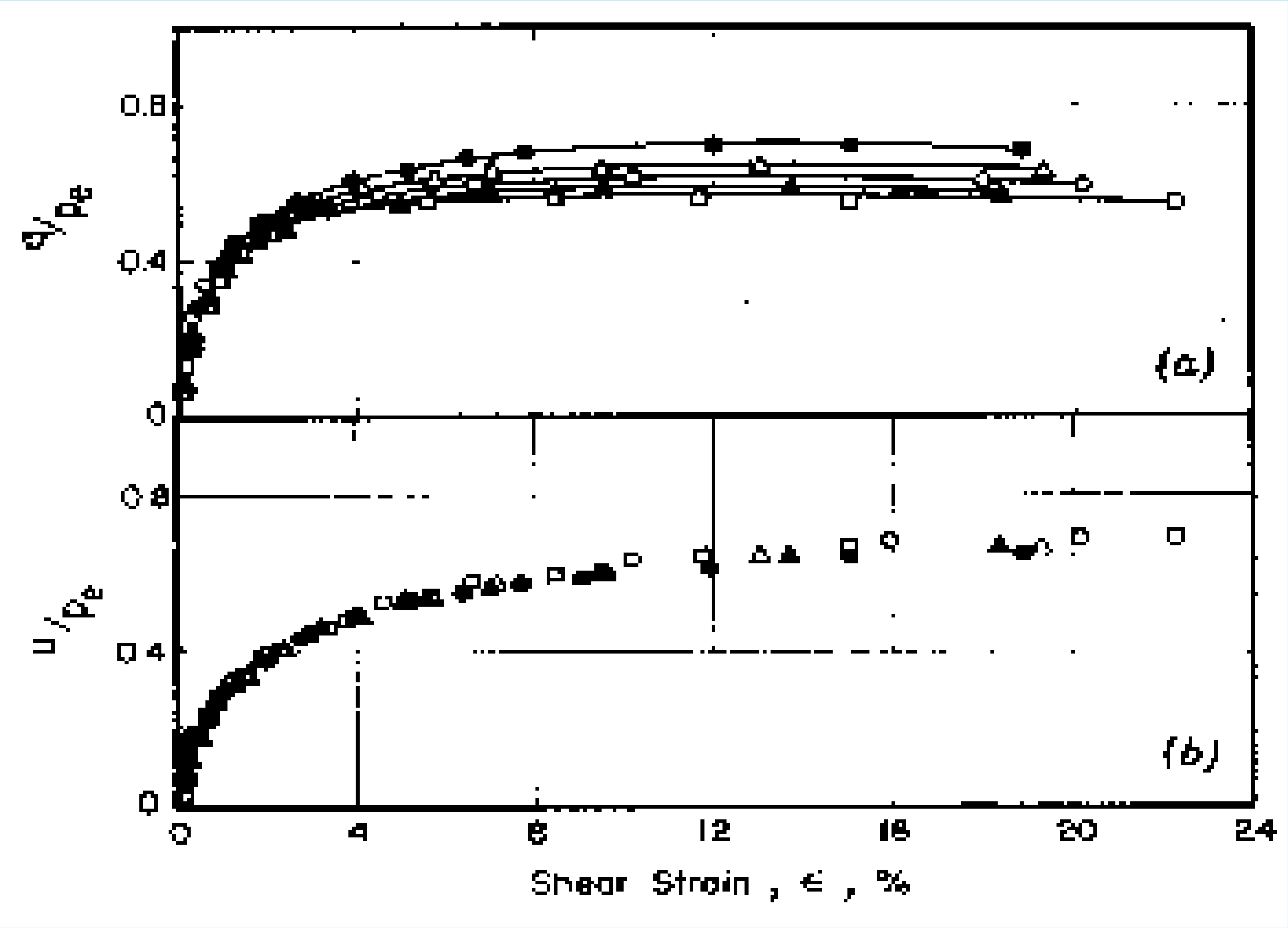


Figure 3.9: Normalized deviator stress and pore pressure relationships

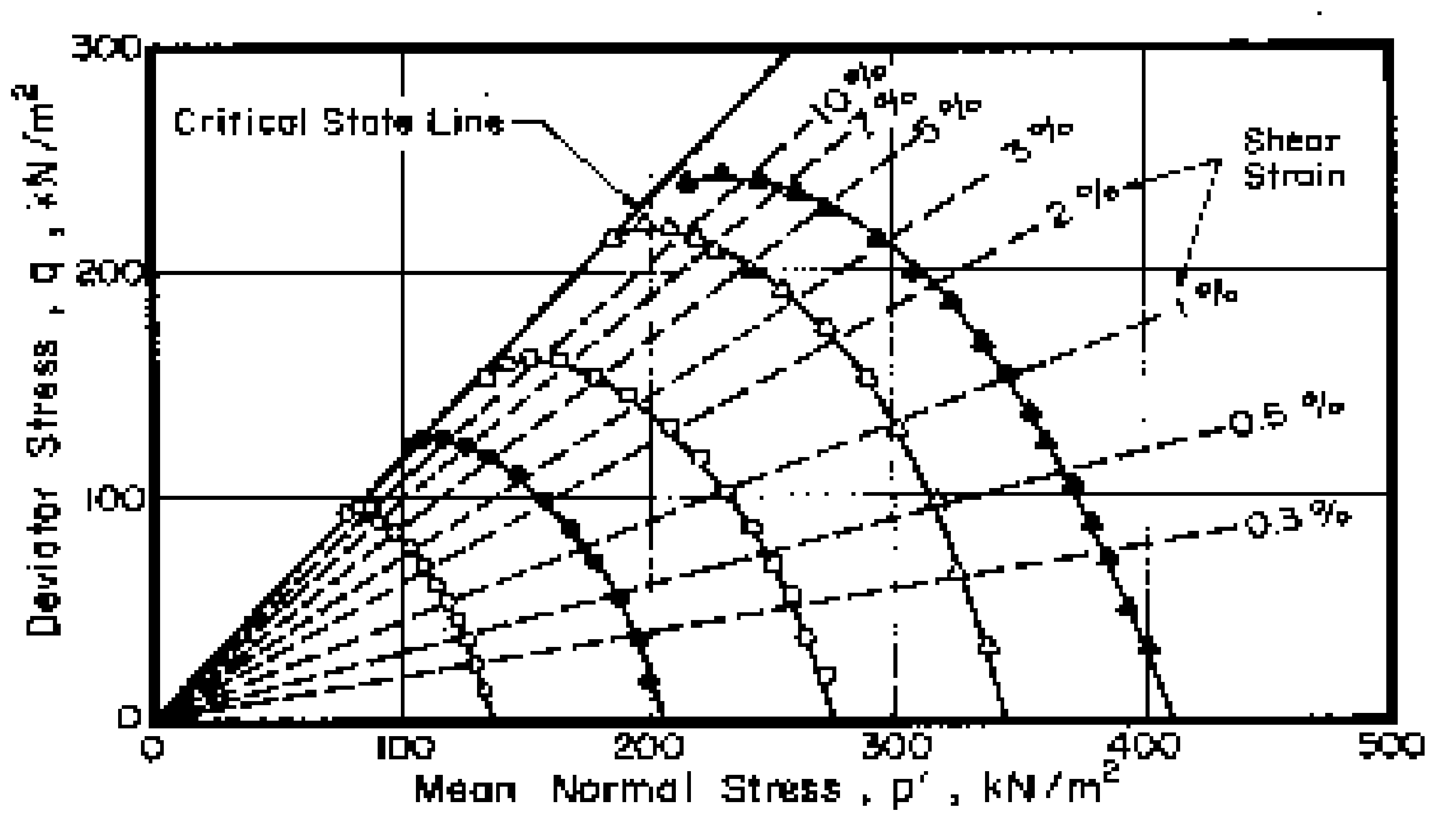


Figure 3.5: Effective stress paths under undrained tests

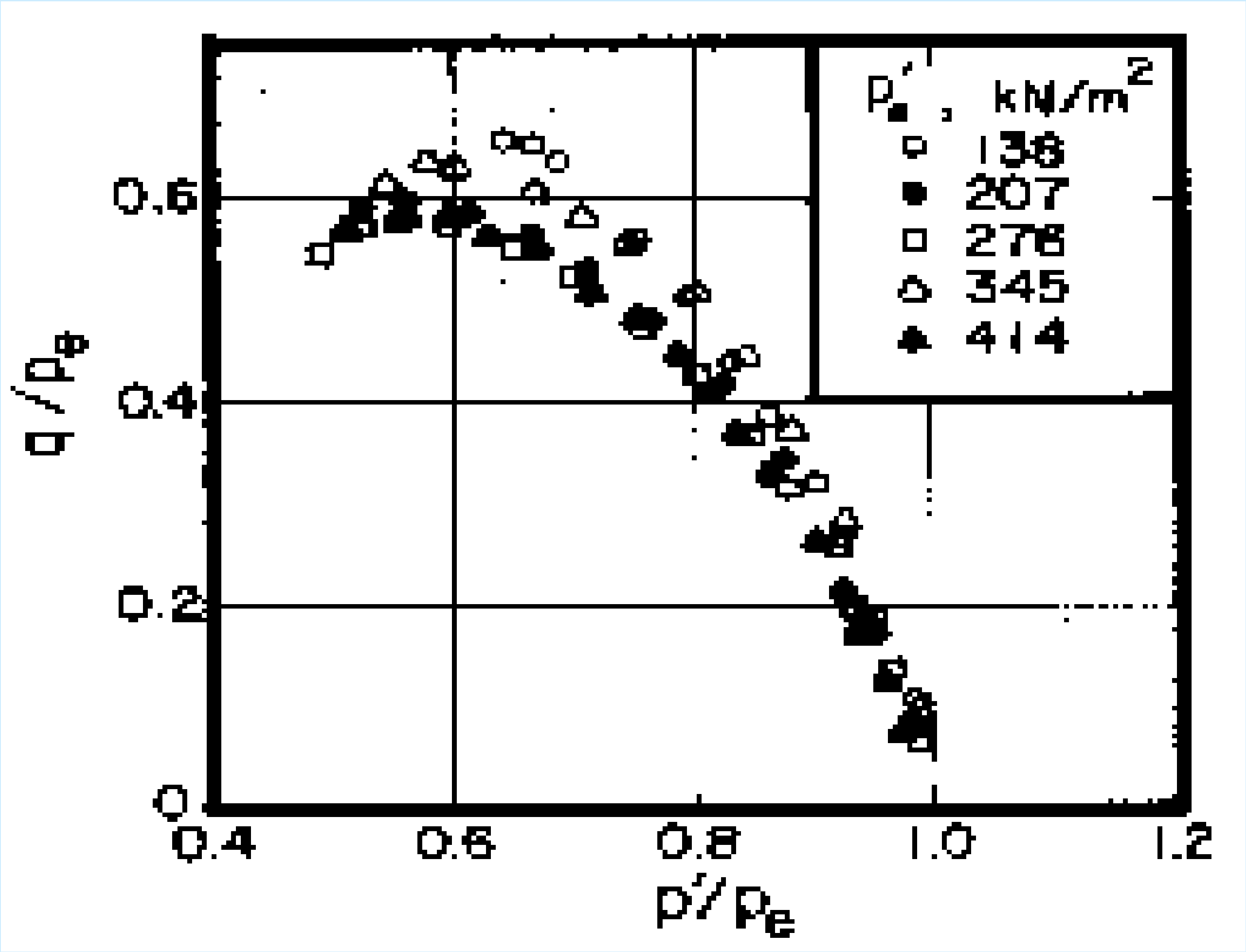


Figure 3.6: Normalized undrained stress paths

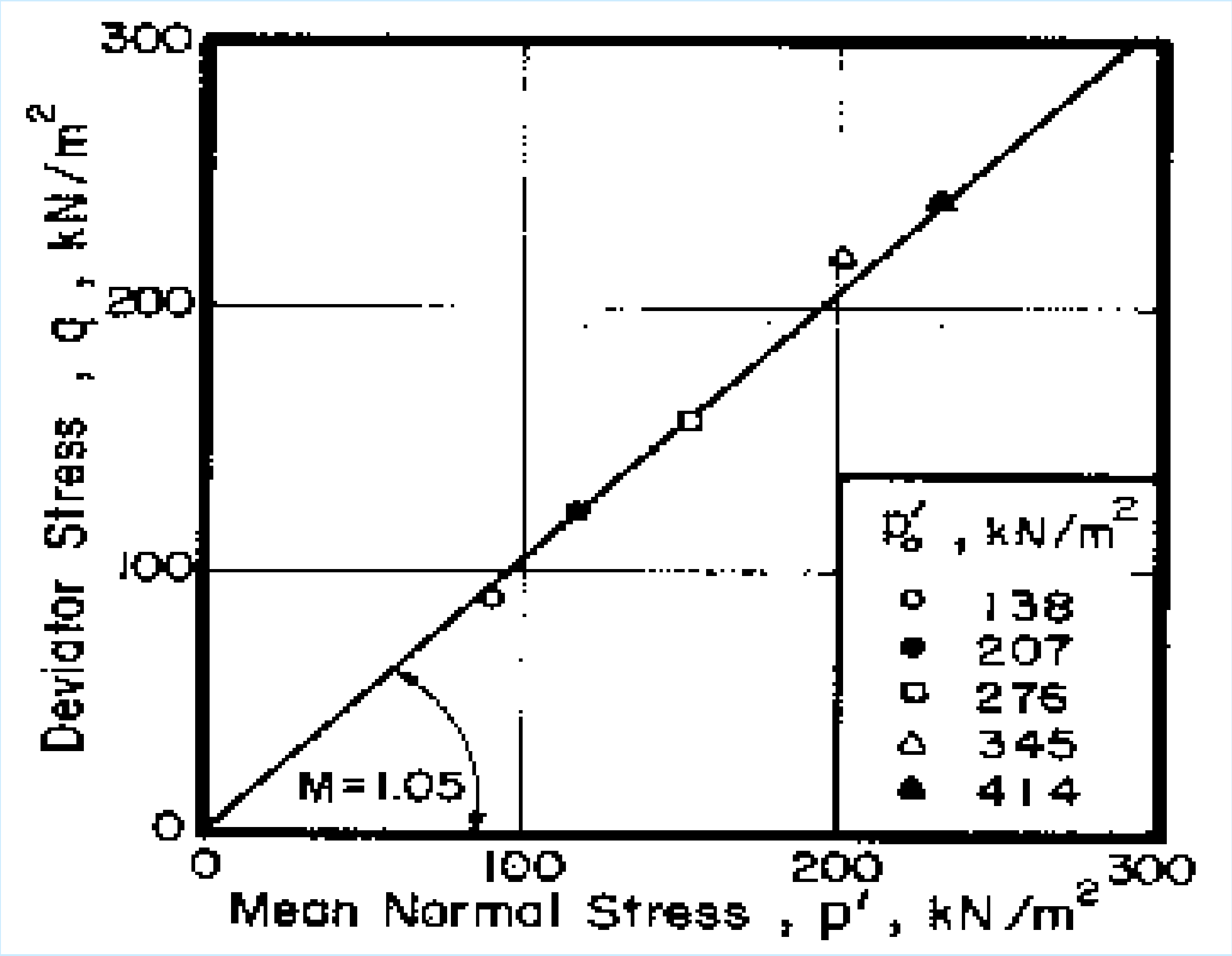


Figure 3.10: Critical state line in (q,p) plot from undrained test data

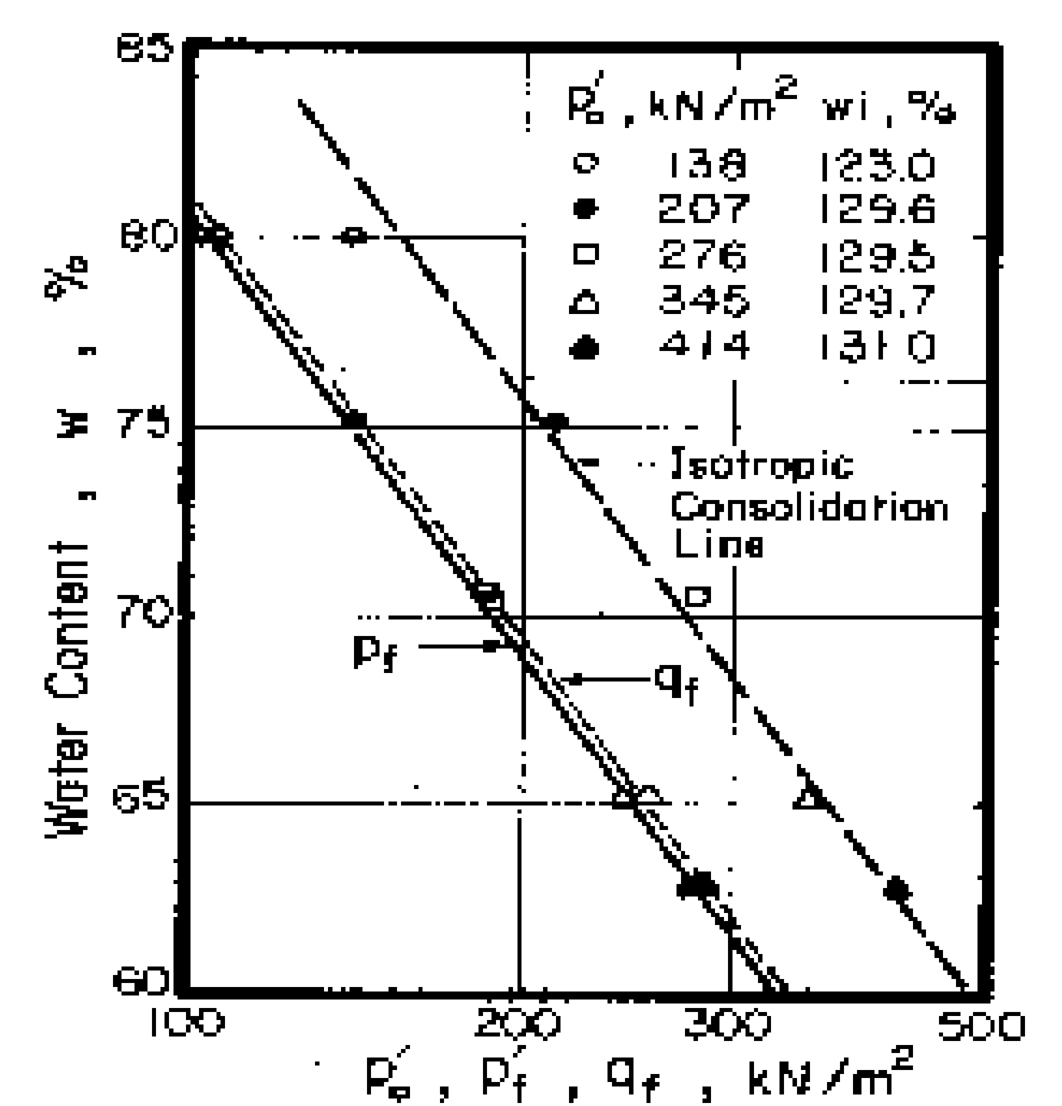


Figure 3.11: Critical state line in (e, ln p) plot

Fully drained behavior

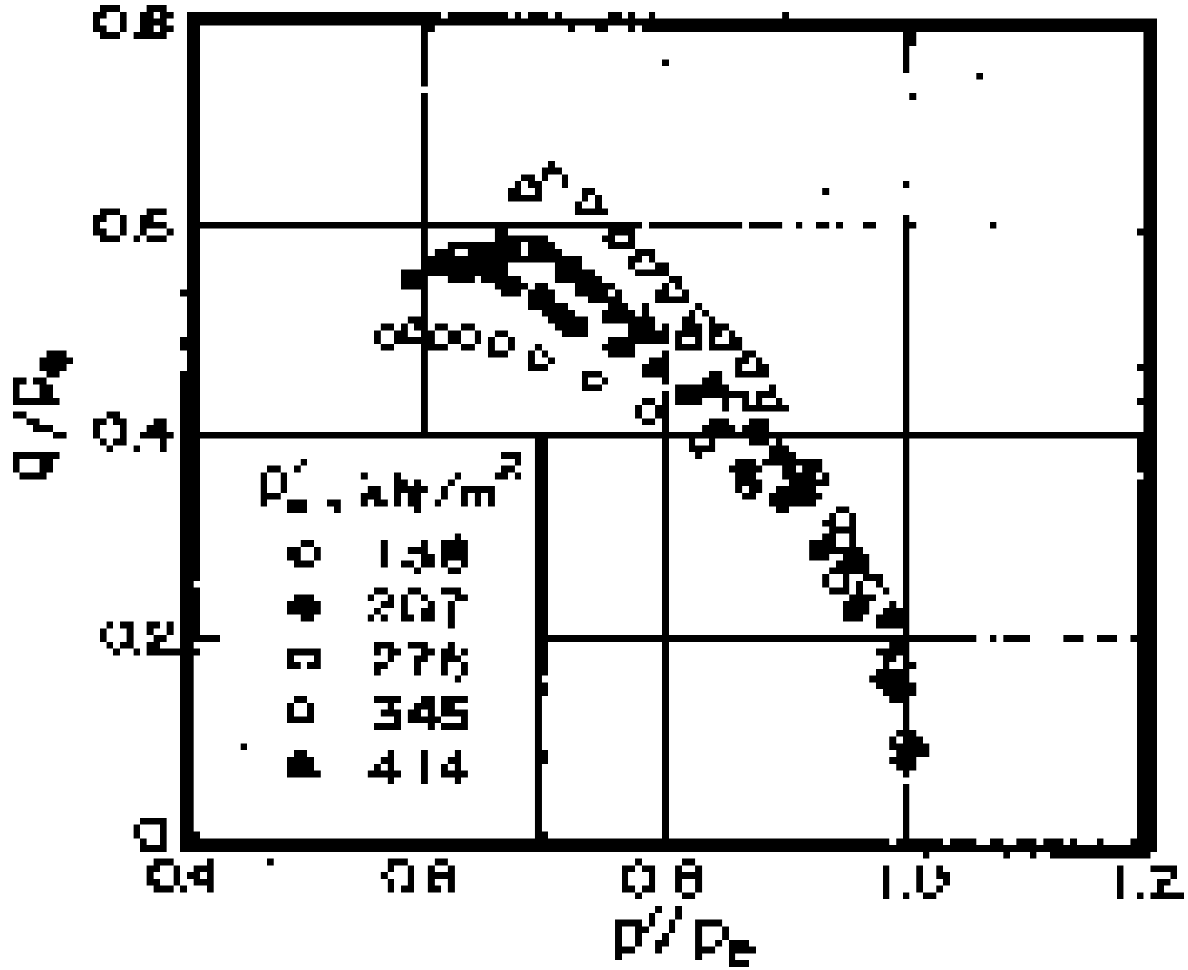


Figure 3.17: State path followed by full drained test specimens

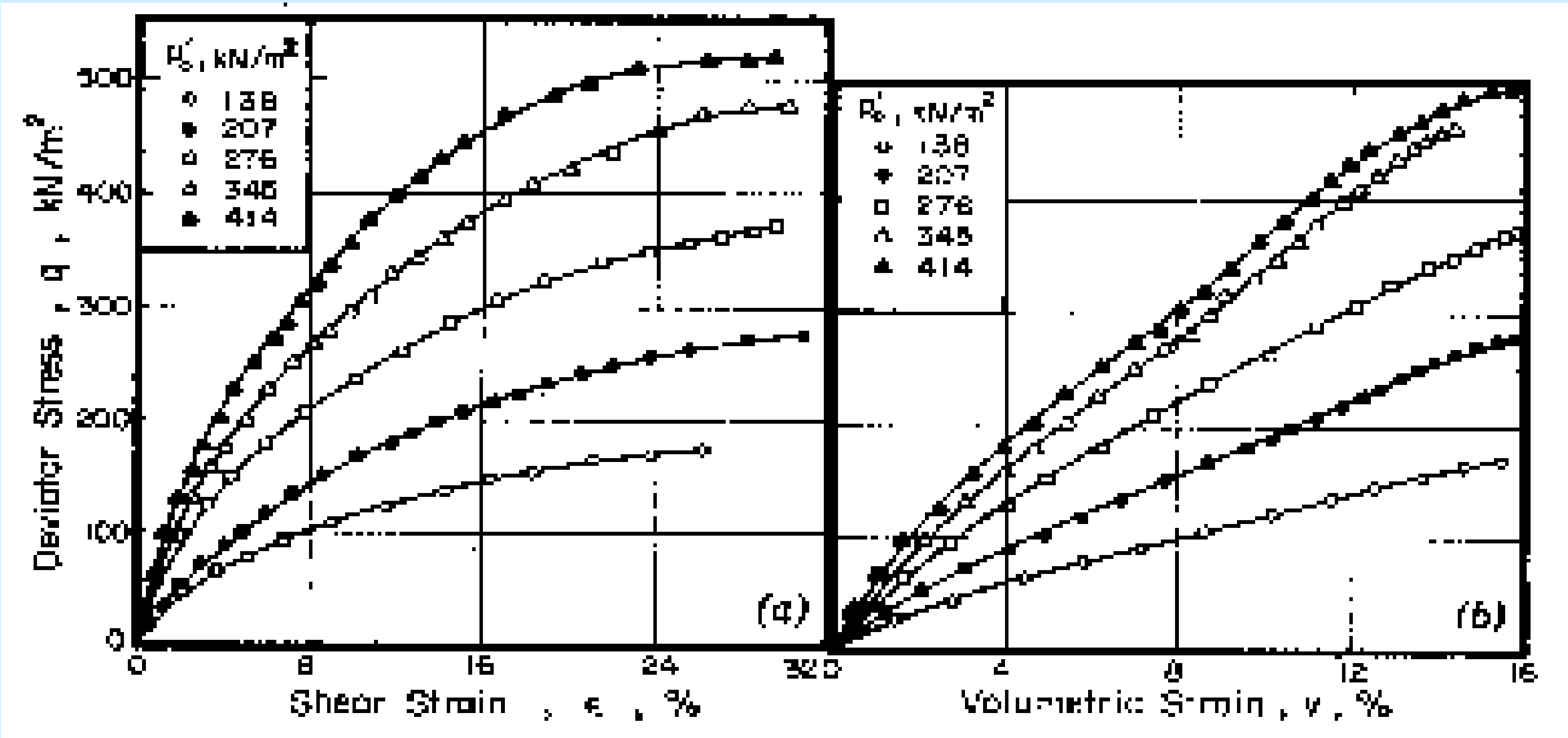


Figure 3.18: Stress strain behavior of full drained test specimens

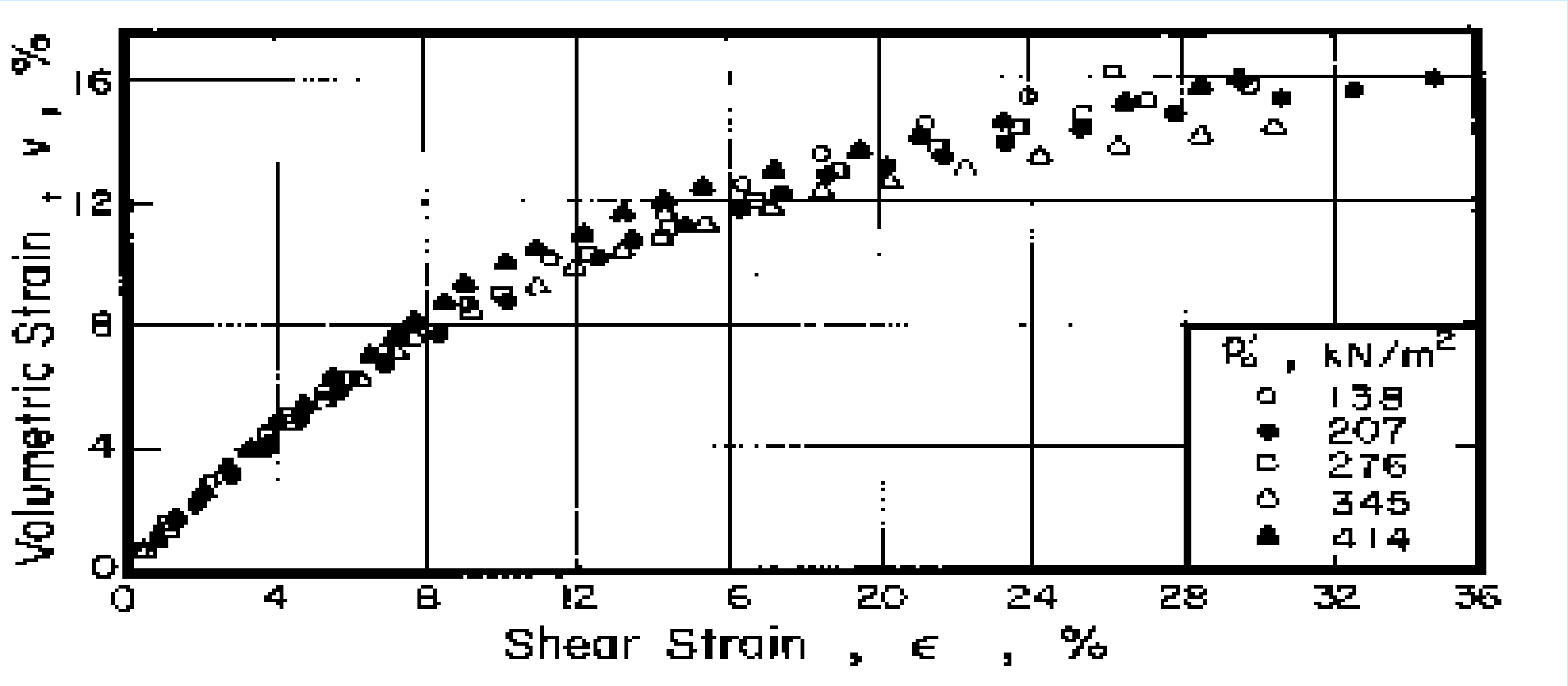


Figure 3.19: Volumetric strain shear strain plot of drained test specimens

The stress ratio strain relationships are shown in Fig. 3.20. The peak stress conditions and the water content log stress relationships are shown in Fig. 3.21 and 3.22. They support the critical state concept.

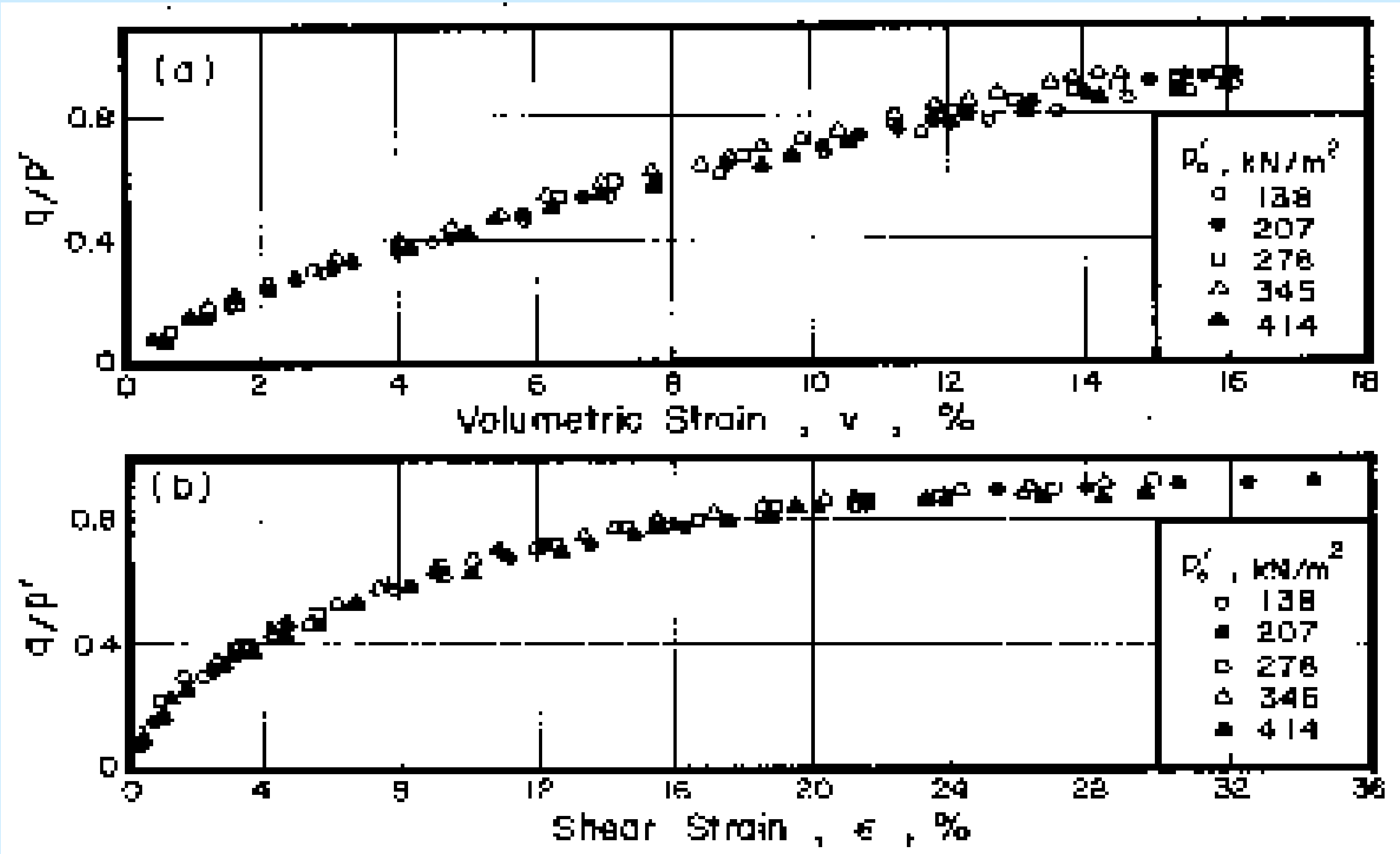


Figure 3.20: Stress ratio strain relationship for fully drained tests

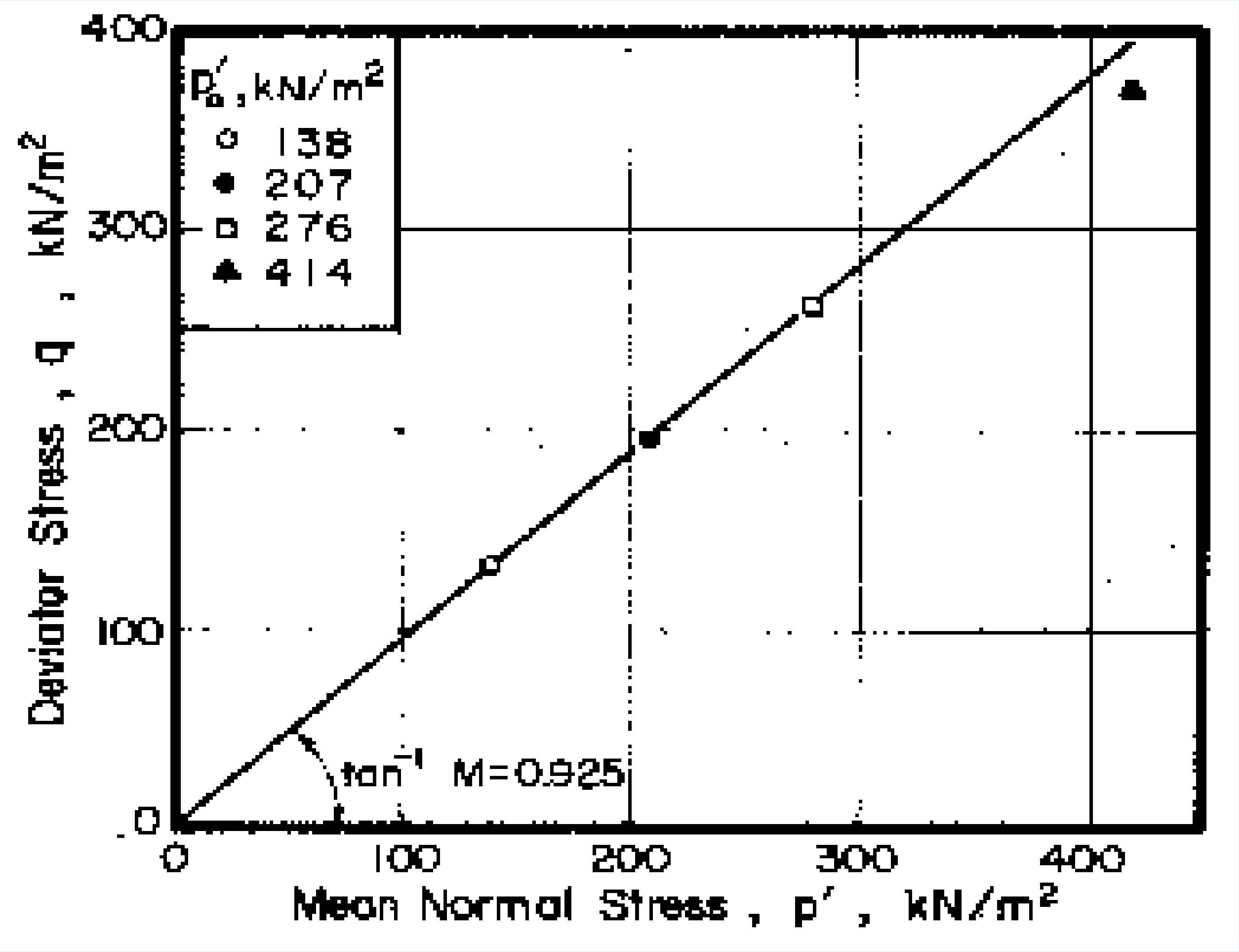


Figure 3.21: Critical state line from constant p and fully drained tests in (q,p) plot

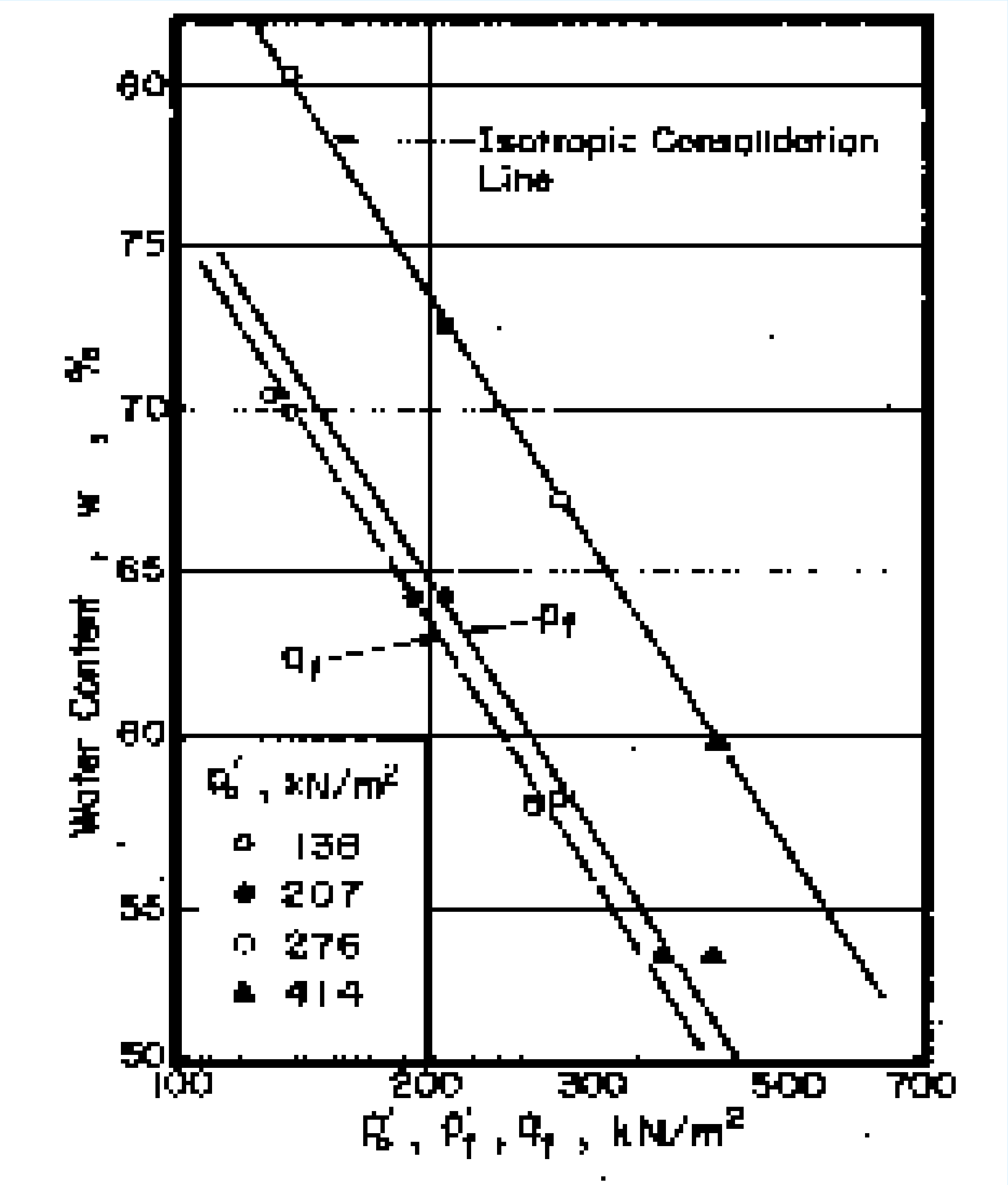


Figure 3.22: Water content Stress projection of critical state line from constant p and fully drained tests

Overconsolidated clay behaviour

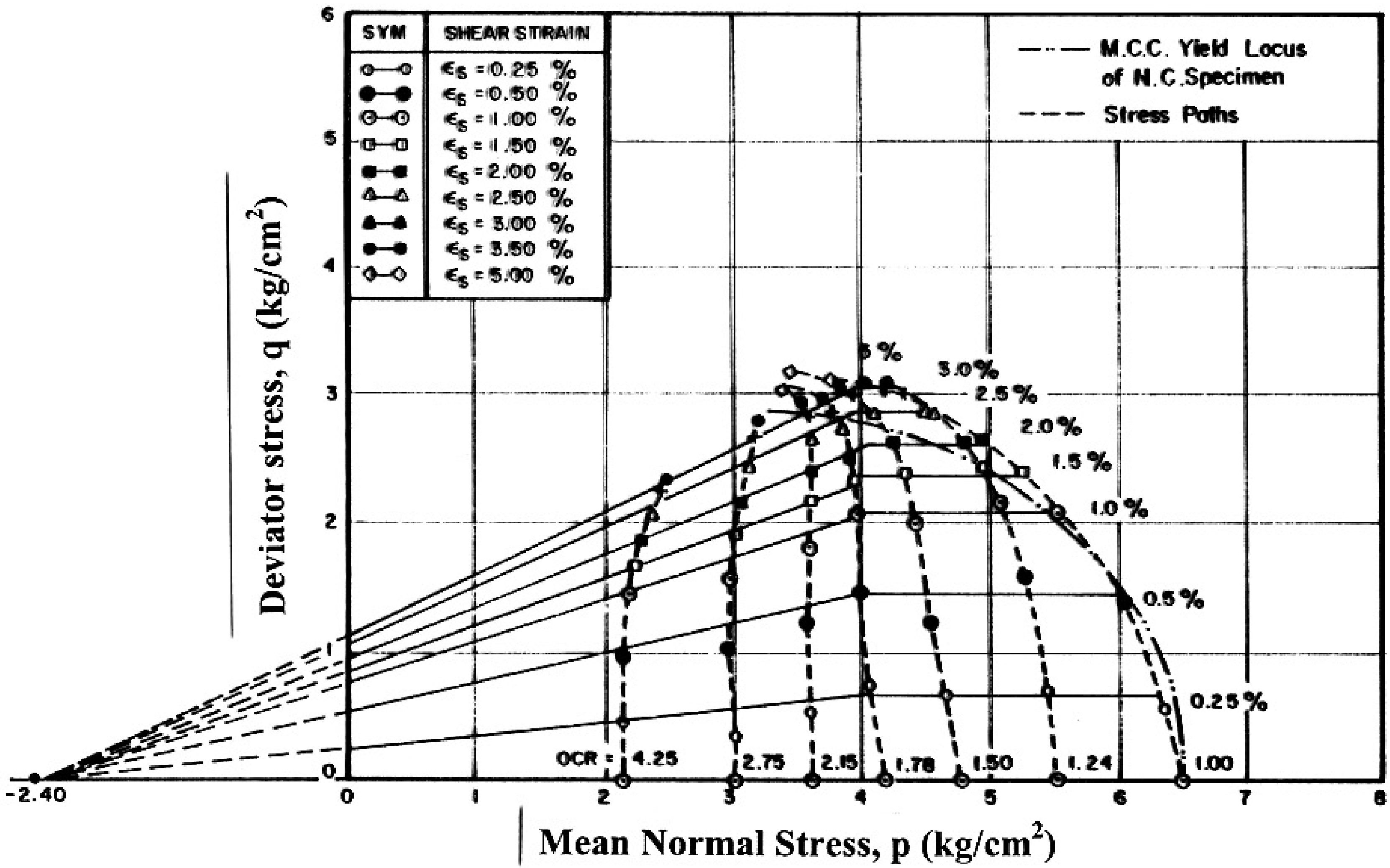


Figure 3.23: Undrained stress paths and constant shear strain contours

Elasto-plastic Behaviour of Soft Clays

Elementary concepts in Elasto-plasticity

Elastic and plastic strains

Elastic strains are strains recovered in a close cycle of stress without any appreciable hysteresis loop. Plastic strains are ir-recoverable strains. The strains due to any increment of stress for an elasto-plastic material are divided into an elastic component and a plastic component.

$$\varepsilon_1 = \varepsilon_1^e + \varepsilon_1^p$$

$$\varepsilon_3 = \varepsilon_3^e + \varepsilon_3^p$$

$$\varepsilon_v = \varepsilon_1 + 2\,\varepsilon_3$$

$$\varepsilon_s = \frac{2}{3}(\varepsilon_1 - \varepsilon_3)$$

Yield surface

An yield surface is defined in the stress space as one which divides the regions of stress for which the strains are elastic from those which include a plastic component. For conditions of stress inside the yield locus an infinitesimally small increment of stress can cause only elastic strains.

If the stress conditions correspond to a point on the yield surface and if the material is stable (as defined by Drucker, 1959), an infinitesimal increment of stress directed outside the yield surface produces only plastic strains for a perfectly plastic material and additional elastic strains if the material work hardens.

Flow rule

The flow rule provides a relationship between the strain rate vector during plastic deformation and the imposed stress vector. As stated before for increment of plastic deformation to occur, the stress point should lie on the yield surface and the stress increment be directed outside the surface.

Formulation of Cambridge Theories

Assuming the slope of the isotropic consolidation line as λ and the isotropic swelling as κ, it can be shown that

$$\frac{d\varepsilon_v}{d\varepsilon_s} = \left[\frac{1}{1-\frac{\kappa}{\lambda}}\right]\left(\frac{M^2-\eta^2}{2\eta}\right)$$

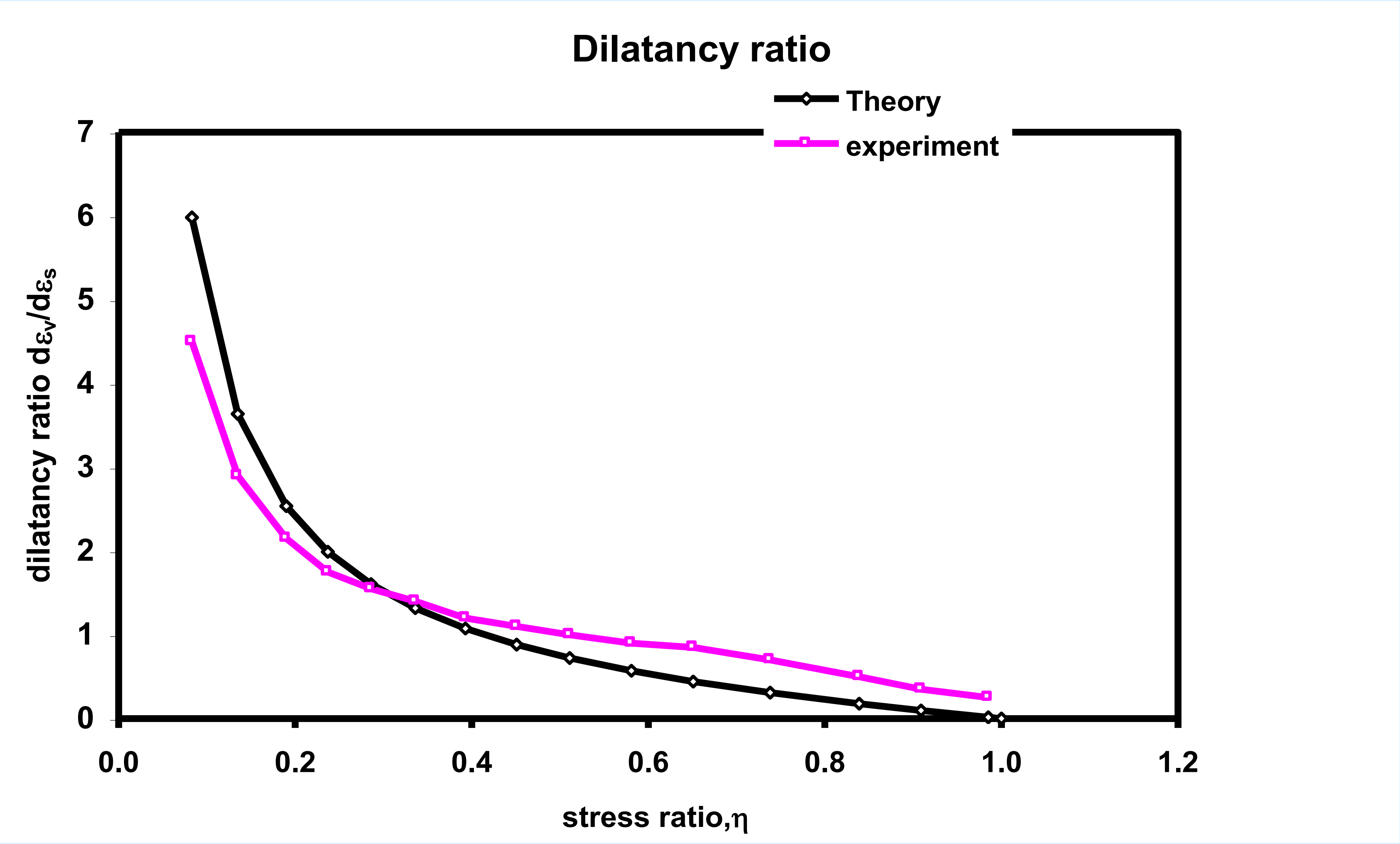


Figure 5.1: Observed and predicted dilatancy ratio

We will then have the equation of the volumetric yield locus as

$$\left(\frac{p}{p_0}\right) = \left(\frac{M^2}{M^2 + \eta^2}\right)$$

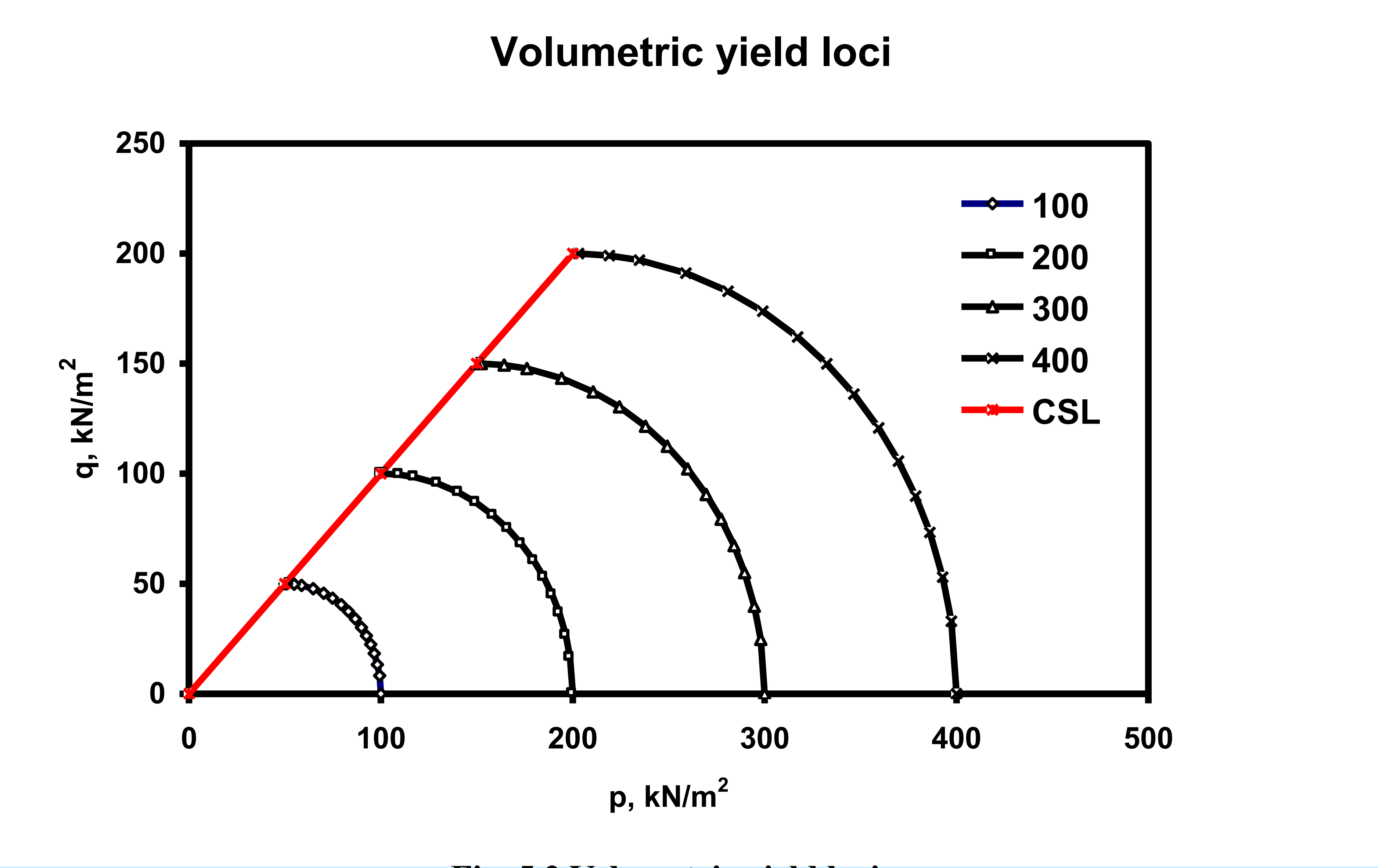


Fig. 5.2 Volumetric yield loci

Incremental volumetric strain

The shifting of the volumetric yield locus will give the plastic volumetric strain increment. Differentiating the volumetric yield locus in equation (5.23), We have

$$\frac{dp_0}{p_0} = \left(\frac{dp}{p} + \frac{2\eta d\eta}{M^2 + \eta^2} \right)$$

Using the consolidation and swelling lines as

$$e_0 - e_a = -\lambda \ln(p_0)$$

$$e - e_0 = -\kappa \ln\left(\frac{p}{p_0}\right)$$

where e_a is the voids ratio under virgin consolidation and at unit pressure.

Differentiating (5.25) and (5.26), We have

$$\left(d\varepsilon_v\right)_{\eta=0} = \left(\frac{\lambda}{1+e_0}\right)\left(\frac{dp_0}{p_0}\right)_{\eta=0}$$

$$\left(d\varepsilon_v{}^e\right) = \left(\frac{\kappa}{1+e_0}\right)\left(\frac{dp_0}{p_0}\right)_{\eta=0}$$

Thus, $$\left(d\varepsilon_v{}^p\right) = \left(\frac{\lambda-\kappa}{1+e_0}\right)\left(\frac{dp_0}{p_0}\right)_{\eta=0}$$

Using (5.24) and (5.29), the incremental plastic volumetric strain can now be obtained as

$$\left(d\varepsilon_v{}^p\right) = \left(\frac{\lambda-\kappa}{1+e}\right)\left(\frac{2\eta d\eta}{M^2+\eta^2}+\frac{dp}{p}\right)$$

The incremental elastic volumetric strain is given by

$$\left(d\varepsilon_v{}^e\right)=\left(\frac{\kappa}{1+e}\right)\left(\frac{dp}{p}\right)$$

Therefore the incremental volumetric strain is

$$d\varepsilon_v=\left(\frac{1}{1+e}\right)\left[(\lambda-\kappa)\left(\frac{2\iota d\eta}{M^2+\eta^2}\right)+\frac{dp}{p}\right]$$

Undrained stress path

Undrained stress path

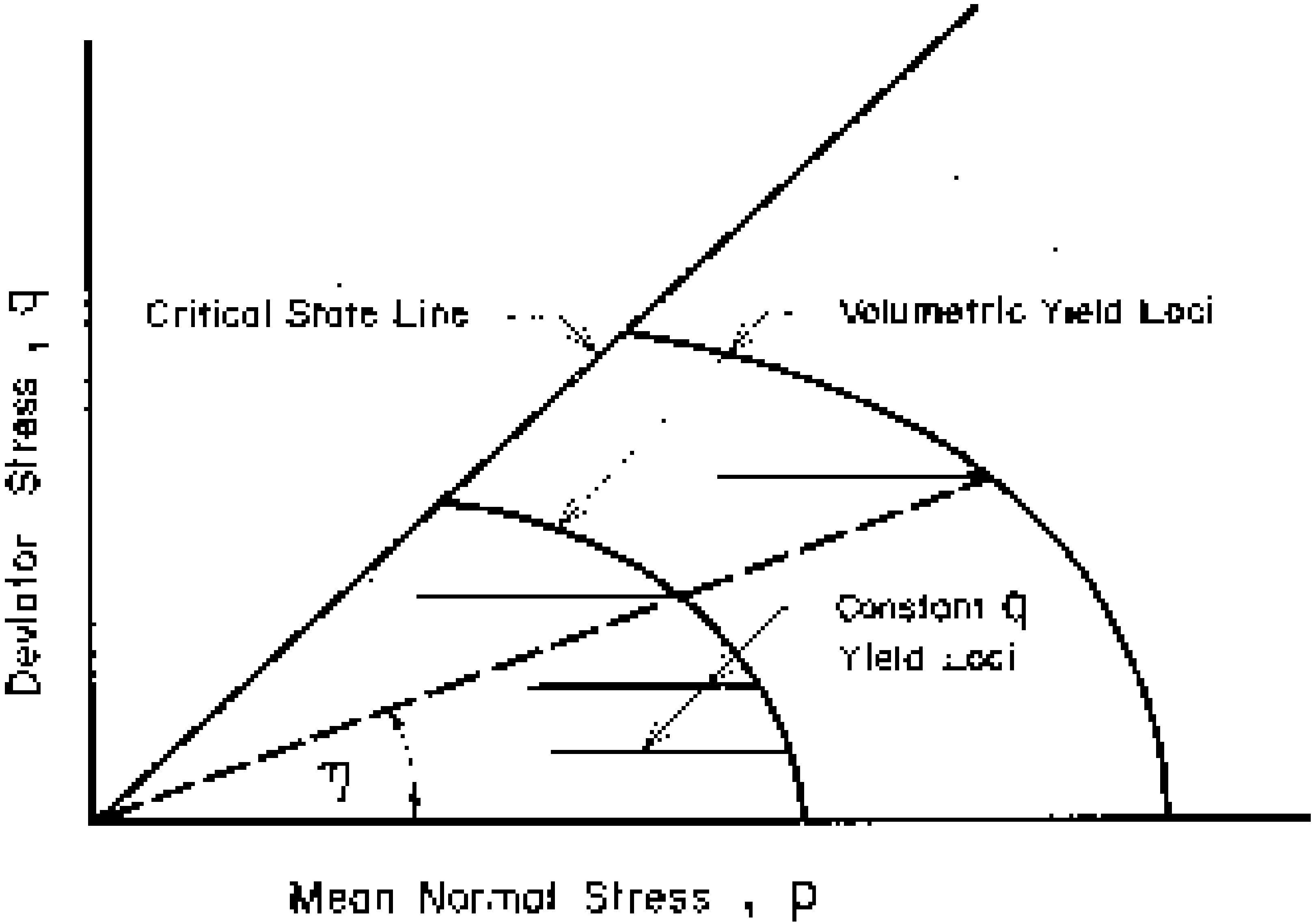
$$\frac{p}{p_0} = \left(\frac{M^2}{M^2 + \eta^2} \right)^{\left(1 - \frac{\kappa}{\lambda}\right)}$$

Shear strain increment

The elastic shear strain increment is assumed to be zero. Therefore the total shear strain increment is the same as the plastic shear strain increment. Therefore,

$$d\varepsilon_s = d\varepsilon_s{}^p = \left(\frac{\lambda - \kappa}{1+e}\right)\left(\frac{2\eta}{M^2 - \eta^2}\right)\left(\frac{2\eta d\eta}{M^2 + \eta^2} + \frac{dp}{p}\right)$$

Use of Double yield loci


Critical State Line
Volumetric Yield Loci
Constant q
Yield Loci
η
Deviator Stress , q
Mean Normal Stress , p

$$\left(d\varepsilon_s\right)_{constq} = \varphi'(\eta)d\eta$$

where dη is the increment in stress ratio under the loading conditions.

Thus the actual shear strain increment is givenin the next slide

$$d\varepsilon_s = d\varepsilon_s{}^p = \left(\frac{\lambda-\kappa}{1+e}\right)\left(\frac{2\eta}{M^2-\eta^2}\right)\left(\frac{2\eta d\eta}{M^2+\eta^2}+\frac{dp}{p}\right) + \varphi'(\eta)d\eta$$

The addition of such shear strain from the constant q yield loci will make the flow rule non- associated in a global sense. The predicted and the observed strains are shown in Fig. 5. 5.

$$\frac{\delta\varepsilon_v{}^p}{\delta\varepsilon_s{}^p} = \frac{(\lambda - \kappa)}{vp\left(M^2 + \eta^2\right)} \begin{bmatrix} \left(M^2 - \eta^2\right) & 2\eta \\ 2\eta & \dfrac{4\eta^2}{M^2 - \eta^2} \end{bmatrix} \begin{bmatrix} \delta p \\ \delta q \end{bmatrix}$$

Pender model for overconsolidated clay

$$d\varepsilon_s{}^p = \frac{2\kappa\left(\dfrac{p}{p_{cs}}\right)\eta d\eta}{M^2(1+e)\left(2\dfrac{p_0}{p}-1\right)\left\{M-\left(\dfrac{p}{p_{cs}}\right)\eta\right\}}$$

$$d\varepsilon_v{}^p = \frac{2\kappa\left(\dfrac{p_0}{p_{cs}}-1\right)\left(\dfrac{p}{p_{cs}}\right)\eta d\eta}{M^2(1+e)\left(2\dfrac{p_0}{p}-1\right)}$$

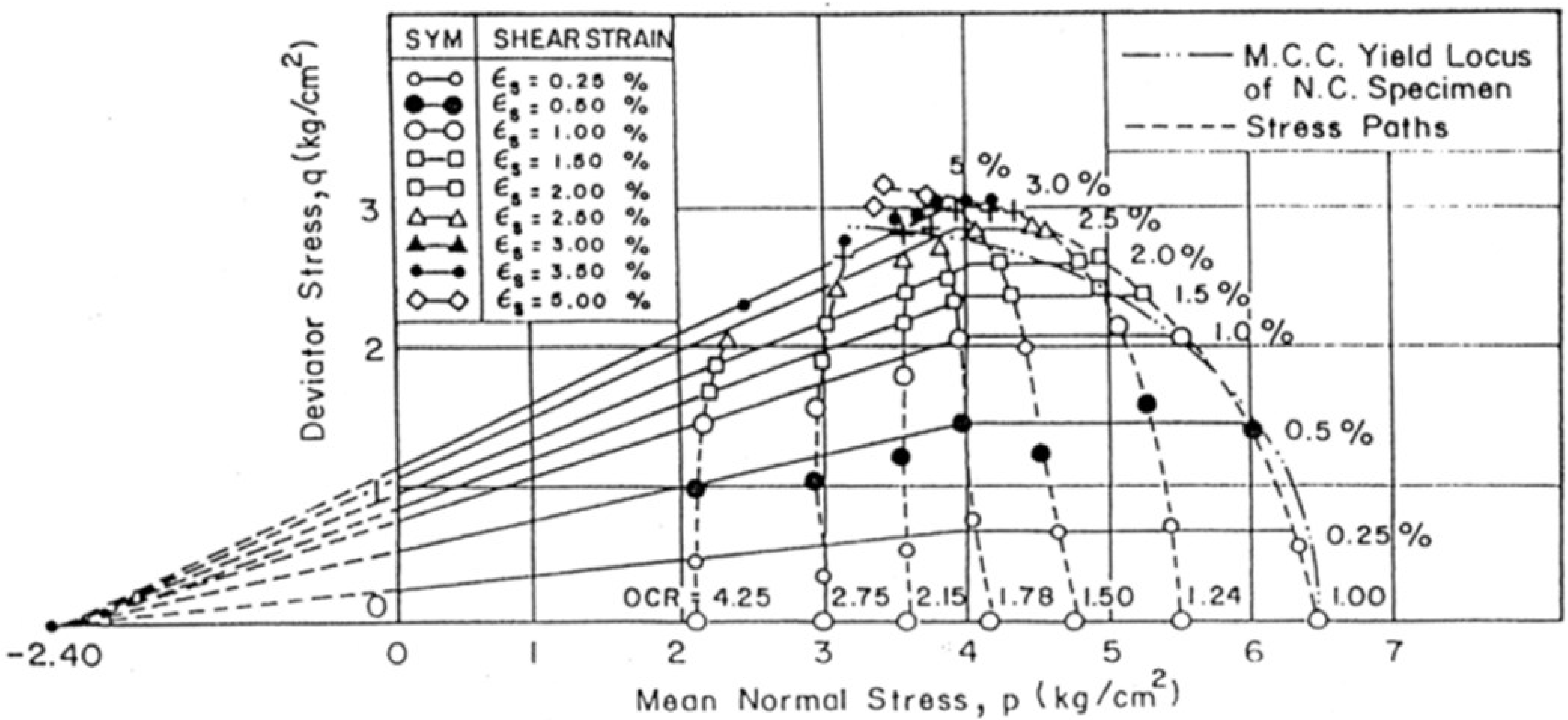


Figure 5.6: Undrained shear strains in overconsolidated clay

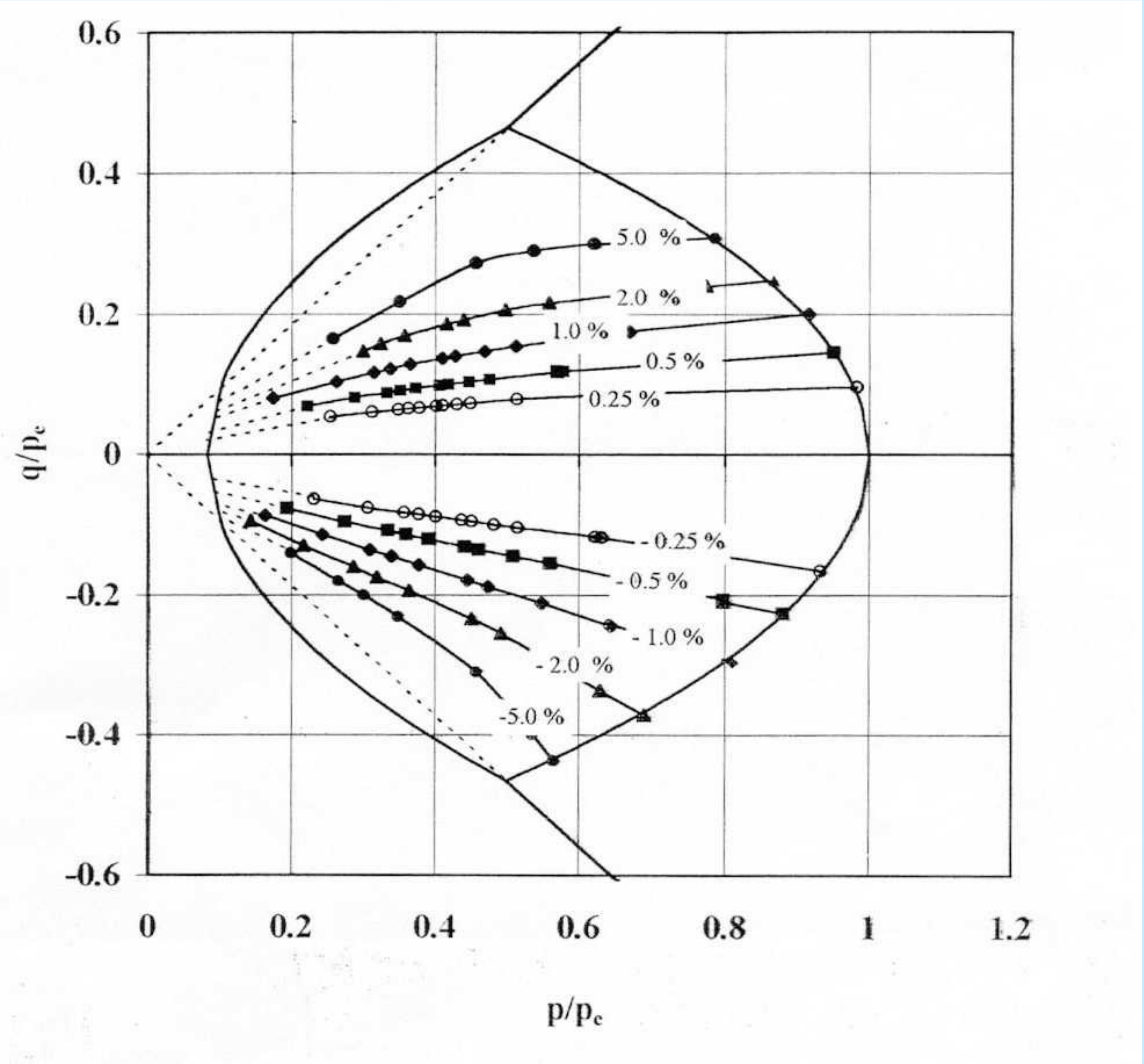


Figure 5.7: Pender prediction of undrained shear strain in overconsolidated clay