# Probability and Statistics Approach for Determining Pore Size Distribution of Coarse-Grain Soil

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**ABSTRACT:** The basic scheme of microscopic mechanical model is proposed to analyze the various mechanical behaviors of unsaturated coarse-grain soil. Physical quantities required for the proposed model are grain size distribution, soil particle density, void ratio and water content. The probability theory and inferential statistics are applied to relate the macroscopic physical quantities used in the conventional soil mechanics to the microscopic physical quantities in the proposed model.

**KEYWORDS:** Unsaturated soil, Microscopic model, Probability, Grain size distribution, Void ratio, Water content.

## 1. INTRODUCTION

The soil above the ground water table is usually considered to be under unsaturated. Unsaturated soil consists of soil particles (solid phase), pore water (liquid phase) and pore air (gas phase), i.e., a three-phases' material, where the shape and size of soil particles are irregular and the shapes and sizes of pores filled with water and air, are inevitably random (see Figure 1). Therefore, the probability theory and inferential statistics are essential to analyze the mechanical behaviors of soil microscopically.

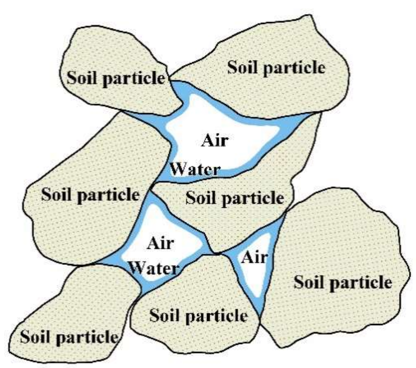



Figure 1 Schematic microscopic state of unsaturated soil on particle scale

It is well known that unsaturated soil has an important physical quantity which is not present in saturated soil which is soil suction that depends on the water content and the void ratio. In additional, unsaturated permeability is a function of water content and void ratio whereas saturated permeability coefficient is only the function of the void ratio.

The mechanical properties of unsaturated soil (three-phase' material) are more complicated, compare with those of saturated soil, a two-phase material consisting of soil particles and pore water. The theoretical research work on soil mechanics during the 20th century focused on the development of saturated soil based on the continuum mechanics. However, it is necessary for the geotechnical engineering field to recognize that unsaturated soil is not a continuous body but a discontinuous body consisting of three-phases, i.e., soil particles (solid), pore water (liquid) and air (gas).

In this paper the probability and statistics approach is proposed to determine the pore size distribution of coarse-grain soil by using the grain size distribution and void ratio, where the coarse-grain soil is defined based on the Japanese soil classification system i.e., more than 50% of the soil particles have diameter larger than 0.075 mm. In this paper, the volume-mass relation of soil is first derived by using the phase diagram and then the microscopic model called the elementary particulate model (EPM) is proposed to identify the state with respect to water content in the saturated-unsaturated soil.

## 2. MACROSCOPIC PHYSICAL QUANTITIES OF SOIL AND PHASE DIAGRAM

Figure 1 shows the schematic microscopic state of unsaturated soil. It is shown in Figure 1 that the soil consists of soil particles (solid phase) and pores (water and gas phases), a three-phase material. In the conventional soil mechanics, the physical quantities which describe the state of soil are defined by using the phase diagram shown in Figure 2(a) where the left side and right side denote the mass and volume of the soil block, respectively. Figure 2(b) shows the alternative phase diagram where the mass and the volume of soil particles are unity, respectively, and Figure 2(c) is the another phase diagram where the volume of soil particle is unity. Therefore the phase diagram shown in Figure 2 can be considered to be one model to express the physical state in a soil block. The air density is about 1/1000 of water density and hence, the mass of air is neglected in Figure 2.

Referring to Figures 2(a), (b) and (c), the macroscopic physical quantities of soil used in the conventional soil mechanics are defined as follows:

Soil particle density: $\rho_s = \dfrac{M_s}{V_s}$ (1)

Void ratio: $e = \dfrac{V_v}{V_s}$ (2)

(Gravimetric) water content: $w = \dfrac{M_w}{M_s}$ (3)

Porosity: $n = \dfrac{V_v}{V} = \dfrac{e}{1+e}$ (4)

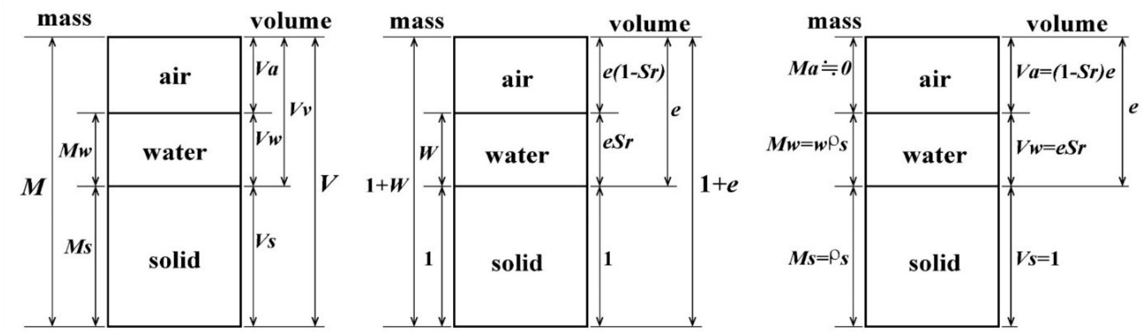
Water porosity (Volumetric water content):

$$n_w = \frac{V_w}{V} = \frac{wG_s}{1+e} \quad (5)$$

Air porosity: $n_a = \dfrac{V_a}{V} = \dfrac{e - wG_s}{1+e}$ (6)

Degree of saturation: $S_r = \dfrac{V_w}{V_v} = \dfrac{wG_s}{e}$ (7)

where $G_s$: Specific gravity of soil particles.




(a) Phase diagram

(b) Phase diagram for unit mass and unit volume of solid phase

(c) Phase diagram for unit volume of solid phase

Figure 2 Phase diagram of soil

These macroscopic physical quantities defined by Equations (2)-(7) for soil are regarded as the arithmetical mean of the microscopic ones in this paper.

The dry and wet densities which can be directly obtained from the laboratory soil tests are related to the soil particle density, void ratio and water content as follows:

Dry density: $\rho_d = \dfrac{\rho_s}{1+e}$ (8)

Wet density: $\rho_t = \dfrac{1+w}{1+e}\rho_s$ (9)

The following equation can be derived by considering the equilibrium of mass of the liquid phase in Figure 2(c).

$$w \cdot \rho_s = \rho_w \cdot e \cdot S_r \quad (10)$$

Equation (10) is rewritten as follows.

$$e \cdot S_r = w \cdot \frac{\rho_s}{\rho_w} = w \cdot G_s \quad (11)$$

Equation (11) is called the volume-mass equitation for soil and the most important equation which expresses the state of unsaturated soil by means of the macroscopic physical quantities, i.e., void ratio, water content and degree of saturation. Equation (11) can be expressed by the distorted surface in 3D space, i.e., $e - w$ - $S_r$ space as shown in Figure 3. Figure 3 shows that the state of soil is limited on the distorted surface.

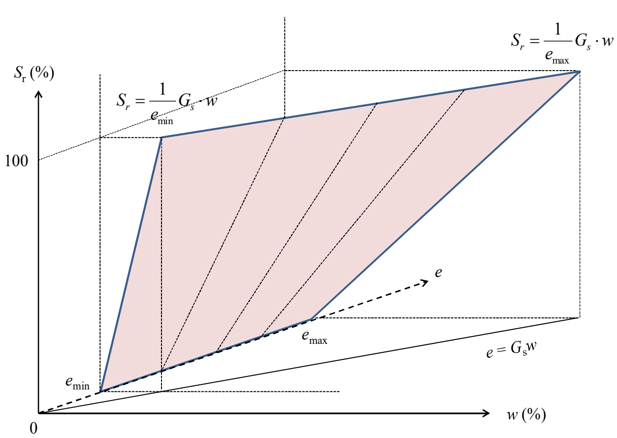



Figure 3 State surface of unsaturated soil

In other words, the state of unsaturated soil is described by Equation (11) and can exist only on the state surface shown in Figure 3. Figure 4 shows the state surface projected on e - w plane, which illustrates that the state of unsaturated soil is limited in the range shown in Figure 4. It can be concluded from Equation (11) that the independent physical quantities of unsaturated soil are two of three physical quantities, i.e., void ratio, water content and degree of saturation for unsaturated soil. In this paper the void ratio and water content are adopted as independent and basic physical quantities for unsaturated soil. Furthermore it is found from Equations (4)-(9) that the macroscopic physical quantities except soil particle density, void ratio and water content can be expressed by void ratio and water content, referring to Figure 2(c). In geotechnical engineering practice, the dry and wet densities of undisturbed and disturbed samples are directly measured in the laboratory and the void ratio and water content are calculated by using Equations (8) and (9).

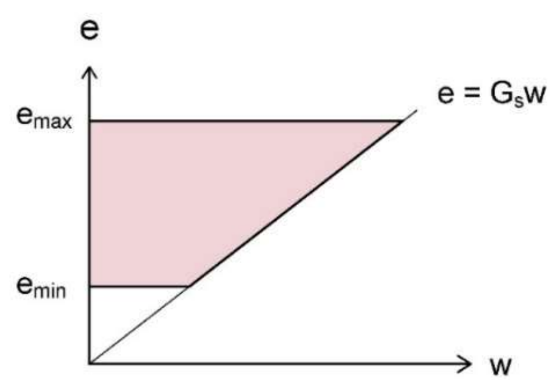



Figure 4 State surface projected on *e*-*w* plane

For saturated soil Equation (11) is rewritten as follows, as the degree of saturation $S_r$ is 1.

$$e = w \cdot G_s \quad (12)$$

Therefore the independent physical quantities of soil are one, i.e., void ratio or water content. In this paper the void ratio is adopted as the independent and basic physical quantity for saturated soil.

Figure 5 shows an example of compression curve of saturated soil (positive pressure) and soil-water characteristic curve (negative pressure) in *e*-*w*-*p* space. The compression curve for saturated soil can be drawn on the vertical plane with $e = wG_s$ when the positive pressure changes. The soil-water characteristic curve can be drawn in the negative pressure space when the water content changes.

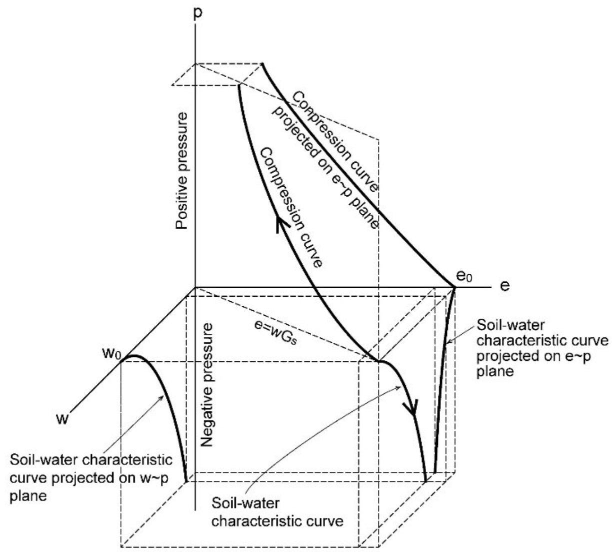



Figure 5 Schematic depiction of mechanical behaviors of saturated and unsaturated soil in 3D space

## 3. MICROSCOPIC MECHANICAL MODEL FOR UNSATURATED SOIL

### 3.1 Elementary particulate box (EPB)

Figure 6(a) shows a triaxial soil specimen, 5 cm in diameter and 10 cm in height. If the specimen with the void ratio of 0.8 is assumed to consist of uniform spheres of 1 mm in diameter, more than $2\times10^5$ particles are present in the specimen. This number is enough to treat the triaxial specimen as a population in the inferential statistics. Hence an imaginary cube taken from the specimen by means of a spoon as shown in Figure 6(b) corresponds to a sample population. Figure 6(c) shows a typical cubic sample with side length $D_{cha}$ arbitrarily taken from the sample population.

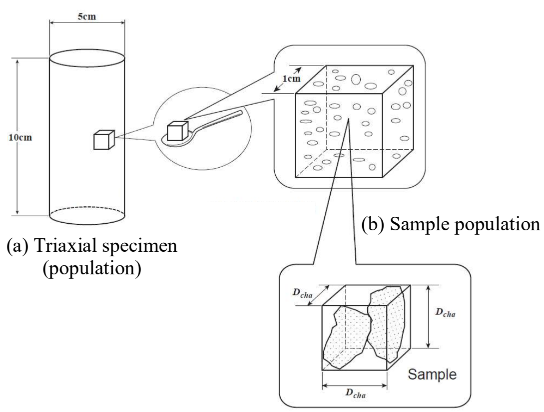



Figure 6 Schematic sample taken from triaxial specimen and its modeling

Here the cubic sample is called the elementary particulate box (EPB) which can be regarded as a sample for the soil particle structure. The side length $D_{cha}$ of EPB (see Eq.(A3.7) ) is called the characteristic length of the soil block. It is found from the numerical calculation that $D_{cha}$ is about $D_{10}$ (diameter finer than 10 % in cumulative grain size distribution curve). The diameter of particle and contact angle formed by adjacent particles at the contact point, and the size and predominant direction of pore of EPB can be considered as the attributes of a sample for the soil particle structure and the pore structure, respectively.

If $D_{cha}$ is assumed to be 0.1 mm, which is common for coarse-grain soil, the number of EPB included in the triaxial specimen shown in Figure 7 is about $2\times10^8$, i.e., when the triaxial specimen and EPB are regarded as the population and the sample, respectively, it means that the population has about $2\times10^8$ samples. Hence, the following equation can be derived by using Equations (A1.1) - (A1.3):

$$\rho_s = \frac{M_s}{V_s} = \frac{\sum_{i=1}^{N} M_{s.i}}{\sum_{i=1}^{N} V_{s.i}} = \frac{\frac{1}{N}\sum_{i=1}^{N} M_{s.i}}{\frac{1}{N}\sum_{i=1}^{N} V_{s.i}} = \frac{E[M_{s,i}]}{E[V_{s,i}]} \tag{13}$$

where $N$: number of samples (i.e. number of EPB),
$M_{s,i}$: mass of soil particles included in the $i$-th EPB,
$V_{s,i}$: volume of soil particles included in the $i$-th EPB,
$E[M_{s,i}]$: mean value of $M_{s,i}$,
$E[V_{s,i}]$: mean value of $V_{s,i}$.

Similarly the basic physical quantities describing the state of soil are obtained from Equation (2) and (3) as follows.

$$e = \frac{V_v}{V_s} = \frac{E[V_{v,i}]}{E[V_{s,i}]} \tag{14}$$

where $V_{v,i}$: volume of soil particles included in the $i$-th EPB.

$$w = \frac{M_w}{M_s} = \frac{E[M_{w,i}]}{E[M_{s,i}]} \tag{15}$$

Where $M_{w,i}$: mass of pore water included in the $i$-th EPB.

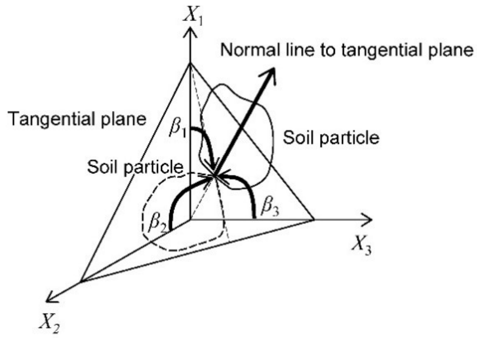
### 3.2 Elementary particulate model (EPM)

A contact point is formed by two adjacent soil particles as shown in Figure 7(a) in 3-D space and the angles between the normal direction to tangential plane and three axes, i.e., a set of ($\beta_1$, $\beta_2$, $\beta_3$) are defined as the contact angle at a contact point. The following equation is derived for the direction cosine:

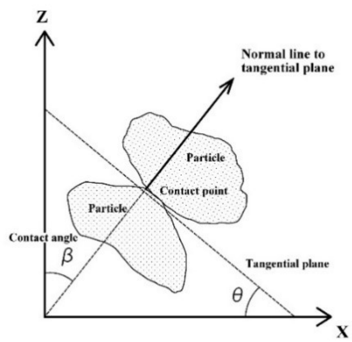
$$\cos^2\beta_1 + \cos^2\beta_2 + \cos^2\beta_3 = 1 \tag{16}$$

The contact angle in 3-D space is defined as a set of ($\beta_1$, $\beta_2$). In the following the consideration is limited to 2-D space and hence, the contact angle is defined as shown in Figure 7(b), i.e., the variable of contact angle is defined as $\beta$.

In the grain size analysis of soil by means of both sieve analysis and sedimentation analysis, the shape of soil particles is assumed to be spherical to estimate the diameter. It means that the grain size distribution curve is obtained by assuming that soil particle is implicitly regarded as a sphere. In the modelling of EPB, a similar way is adopted, i.e., soil particles are assumed to be spherical in shape and in additional, the soil particles are assumed to be rigid. As the soil particles are assumed to be rigid, the grain size distribution is not changed due to particle crushing and/or abrasion at contact points of soil particles in the deformation process of the soil mass.




(a) 3-D space




(b) 2-D space

Figure 7 Definition of contact angle

Figure 8(a) shows the probabilistic model for soil particle structure, where the diameter of soil particle, $D_s$, and the contact angle, $\beta$, at the contact point of soil particles are adopted as independent random variables.

Figure 8 (b) shows the probabilistic model for pore structure, where the pore in EPB is summed up to be a pipe with the diameter, $D_v$, with the inclination angle, $\theta$, which is assumed to be the predominant flow direction of pore water and pore air. Thus, $D_v$ and $\theta$ are adopted as random variables for pore structure. The other parts of the EPB are regarded as the impermeable solid. The modelling procedure is that same as that of phase diagram derived from the real soil block.

The models for soil particle structure and pore structure shown in Figure 8 are collectively called the elementary particulate model in this paper.

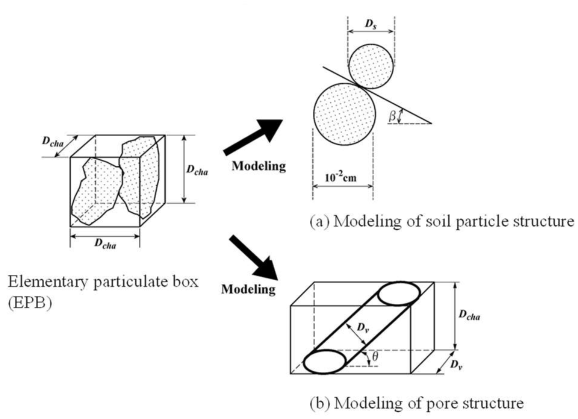



Figure 8 Modelling of soil particle structure and pore structure in EPB

### 3.2.1 Modelling of Soil Particle Structure

It is well known that the grain size distribution curve of most coarse non-organic soil can approximately be expressed by the logarithmic normal distribution. Hence, the cumulative grain size distribution can be expressed by using the probability density function as follows:

$$f_S(D_S) = \frac{1}{\sqrt{2\pi}\cdot\zeta_S\cdot D_S}\exp\left[-\frac{(\ln D_S-\lambda_S)^2}{2\zeta_S^2}\right] \tag{17}$$

where $D_s$: diameter of soil particle assumed to be spherical,
$\lambda_s$: mean value of ln $D_s$,
$\zeta_s$: standard deviation of ln $D_s$.

$$\lambda_S = \ln\mu_S - \frac{1}{2}\zeta_S{}^2 \tag{18}$$

$$\zeta_S{}^2 = \ln\left(1+\frac{\sigma_S{}^2}{\mu_S{}^2}\right) \tag{19}$$

where $\mu_s$: mean value of $D_s$,
$\sigma_s$: standard deviation of $D_s$.

The values of distribution parameter in Equation (17), $\lambda_s$ and $\zeta_s$, are obtained from the grain size analysis in the laboratory test and then $\mu_s$ and $\sigma_s$ can be calculated by using Equation (18) and (19). Note that $\lambda_s$ corresponds to ln $D_{50}$ and $\mu_s$ corresponds to the mean diameter of soil particles, i.e., $D_{50}$, the diameter finer than 50%, is not the mean diameter.

The probability density function of contact angle $\beta$ is assumed to be pentagonal shape as shown in Figure 9 is expressed as follows:

For $-\pi/2 \le \beta \le 0$
$$f_\beta(\beta) = \frac{2/\pi - 2\cdot\varsigma_c}{\pi/2}\cdot\beta + \frac{2}{\pi} - \varsigma_c \tag{20a}$$

For $0 \le \beta \le \pi/2$
$$f_\beta(\beta) = -\frac{2/\pi - 2\cdot\varsigma_c}{\pi/2}\cdot\beta + \frac{2}{\pi} - \varsigma_c \tag{20b}$$

Where $\zeta_c$: distribution parameter which describes the pentagonal shape.

Equation (20) means that the tangential plane at the contact point tends to horizontal under gravity field. Here the ratio of height at $\beta=\pm\pi/2$ to $\beta=0$ is tentatively assumed to be 1:3 referring to the measuring results of contact angle by the microscope (Oda (1972)). The distribution parameter $\zeta_c$ is calculated to be a constant value of 0.159 because $\zeta_c$ is not sensitive to the final results.

Consequently the probabilistic state of soil particle structure can be estimated by Equations (17) and (20).

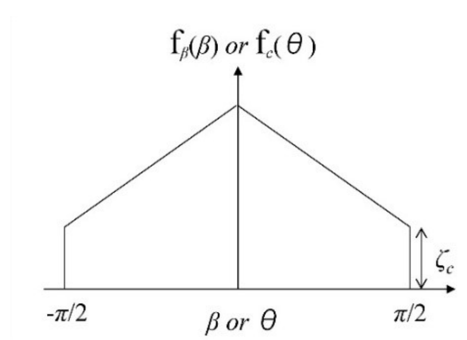



Figure 9 Probability density function of contact angle $\beta$

### 3.2.2 Modelling of Pore Structure

Figure 10 shows the elementary particulate model with arbitrary random variables $D_v$ and $\theta$ which are independent of each other.

Assuming that pore size in soil can also be expressed by the logarithmic normal distribution, the following equation is obtained.

$$f_v(D_v)=\frac{1}{\sqrt{2\pi}\zeta_v\cdot D_v}\exp\left\{-\frac{(\ln D_v-\lambda_v)^2}{2\zeta_v^2}\right\} \tag{21}$$

where $D_v$: diameter of pore assumed to be circular,
$\lambda_v$: mean value of ln $D_v$,
$\zeta_v$: standard deviation of ln $D_v$.

$$\lambda_v=\ln\mu_v-\frac{\zeta_v^2}{2} \tag{22}$$

$$\zeta_v^2=\ln\left(1+\frac{\sigma_v^2}{\mu_v^2}\right) \tag{23}$$

where $\mu_v$: mean value of $D_v$,
$\sigma_v$: standard deviation of $D_v$.

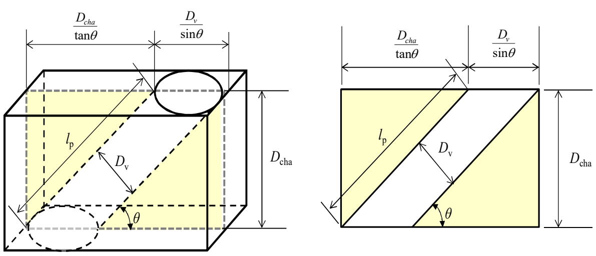



Figure 10 Elementary particulate model (EPM)

Furthermore, assuming that the coefficient of variation is same for soil particle structure and pore structure which means the parallel movement of grain size distribution as shown in Figure 11, the following equation is obtained:

$$\kappa=\frac{\sigma_s}{\mu_s}=\frac{\sigma_v}{\mu_v} \tag{24}$$

As described in the previous subsection, $\mu_s$ and $\sigma_s$ are known from the cumulative grain size distribution curve and thus $\kappa$ is also known. Thus, the only unknown parameter in Equation (21) becomes either $\lambda_v$ or $\zeta_v$ through Equations (22) and (23).

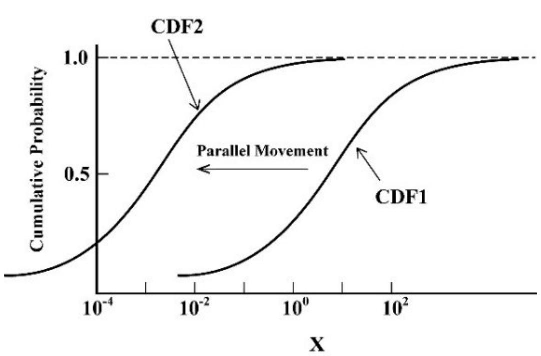



Figure 11 Two cumulative distribution functions, CDF1 and CDF2 with same coefficient of variation and different mean values

In Figure 7(b) it is considered that the flow of pore fluid in the direction of tangential plane at a contact plane is more than the normal to tangential plane. It is found that $\theta$ is equal to $\beta$. Therefore the inclination angle of pipe, $\theta$, which is defined as the predominant flow direction of pore fluid, is assumed to be same as the average of the contact angle in EPB as shown in Figure 10, i.e. the probability density function of predominant flow direction is same as Figure 10. This consideration is similar to the real state of soil as shown in Figure 1 and is modelled by the phase diagram shown in Figure 2. Therefore the probability density function of the predominant flow direction is same as Figure 9 and is expressed as follows:

$$\text{For } -\pi/2\le\theta\le 0 \quad f_c(\theta)=\frac{2/\pi-2\cdot\varsigma_c}{\pi/2}\cdot\theta+\frac{2}{\pi}-\varsigma_c \tag{25a}$$

$$\text{For } 0\le\theta\le\pi/2 \quad f_c(\theta)=-\frac{2/\pi-2\cdot\varsigma_c}{\pi/2}\cdot\theta+\frac{2}{\pi}-\varsigma_c \tag{25b}$$

The volume of elementary particulate model (EPM), $V_{EPM}$ and the volume of pipe in EPM, $V_{EPM,p}$ (corresponding to $V_{s,i}+V_{v,i}$ and $V_{v,i}$ in Equation (14)) in Figure 11 can be expressed by the functions of two random variables $D_v$ and $\theta$, as follows:

$$V_{EPM}=D_v\cdot\left(\frac{D_v}{\sin\theta}+\frac{D_{cha}}{\tan\theta}\right)\cdot D_{cha}=\varphi_{EPM}(D_v,\theta) \tag{26}$$

$$V_{EPM,p}=\pi\cdot\left(\frac{D_v}{2}\right)^2\cdot\frac{D_{cha}}{\sin\theta}=\varphi_{EPM,p}(D_v,\theta) \tag{27}$$

where $\varphi_{EPM}(D_v,\theta)$ and $\varphi_{EPM,p}(D_v,\theta)$: functions corresponding to Eq. (A10) for volume of EPM and pipe.

The volume of the solid part in EPM shown in Figure 10 is obtained using Equations (26) and (27) as follows:

$$V_{EPM,s}=\varphi_{EPM}(D_v,\theta)-\varphi_{EPM,p}(D_v,\theta)=\varphi_{EPM,s}(D_v,\theta) \tag{28}$$

where $V_{EPM,s}$: volume of solid part in EPM.

Using Equations (14), (21), (25), (26), (27), (28) and (A1.12), the following equation can be obtained:

$$e=\frac{V_v}{V_s}=\frac{E[V_{v,i}]}{E[V_{s,i}]}=\frac{E[V_{EPM,p}]}{E[V_{EPM,s}]}=\frac{\int_0^\infty\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\varphi_{EPM,p}(D_v,\theta)\cdot f_v(D_v)\cdot f_c(\theta)d\theta dD}{\int_0^\infty\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\varphi_{EPM,s}(D_v,\theta)\cdot f_v(D_v)\cdot f_c(\theta)d\theta dD} \tag{29}$$

As the void ratio $e$ on the left side of Equation (29) is obtained from the laboratory test, the unknown parameter, either $\lambda_v$ or $\zeta_v$ in the probability density function shown by Equation (21) can be back-calculated numerically.

In the drying process of saturated soil the entry of air into pores filled with water begins in larger pores and inversely in the wetting process of unsaturated soil the entry of water into pores filled with air begins in smaller pores. Considering these behaviours in the elementary particulate model, it can be assumed that the pipe with the range of diameter $0<D_v\leqq d_w$ is filled with water and the pipe with the range of diameter $d_w<D_v<\infty$ is filled with air. Then the following equation can be derived by using Equations (15), (21), (25), (26), (27), (28) and (A1.12), similar to the derivation of Eq. (29):

$$w=\frac{M_w}{M_s}=\frac{E\lfloor M_{EPM,p}\rfloor}{E\lfloor M_{EPM,s}\rfloor}=\frac{\rho_w\cdot E\lfloor V_{EPM,p}\rfloor}{\rho_s\cdot E\lfloor V_{EPM,s}\rfloor}$$

$$=\frac{\rho_w\cdot\int_0^{d_w}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\varphi_{EPM,p}(D_v,\theta)\cdot f_v(D_v)\cdot f_c(\theta)d\theta dD}{\rho_s\cdot\int_0^{\infty}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\varphi_{EPM,s}(D_v,\theta)\cdot f_v(D_v)\cdot f_c(\theta)d\theta dD} \tag{30}$$

As the gravimetric water content $w$ on the left side of Equation (30) is obtained from the laboratory test, the unknown $d_w$ limit of the integral range which corresponds to the boundary between liquid phase and gas phase in the phase diagram shown in Figure 2 can be numerically back-calculated.

Figure 12 shows the procedure to relate the macroscopic physical quantities, grain size distribution, void ratio and water content which can be obtained from the laboratory soil tests to the microscopic physical quantities, pore size distribution, $d_w$. Figure 13 shows the grain size distribution and pore size distribution calculated by Equations (17) and (21).

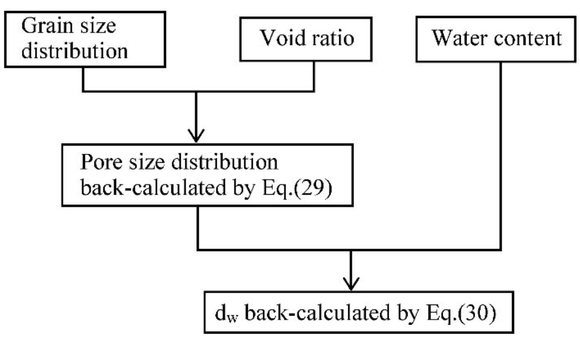



Figure 12 Procedure to relate macroscopic physical quantities to microscopic ones

The grain size distribution as shown in Figure 13 and void ration ($e$=1.46) were used as input parameters. The parameter $d_w$ can be derived from Eq.(30) using pore size distribution curve. The parameter $d_w$ will then be used to calculate suction and permeability coefficient (Sako and Kitamura, 2006).

Finally the list of assumptions to derive Equations (29) and (30) is shown in Table 1.

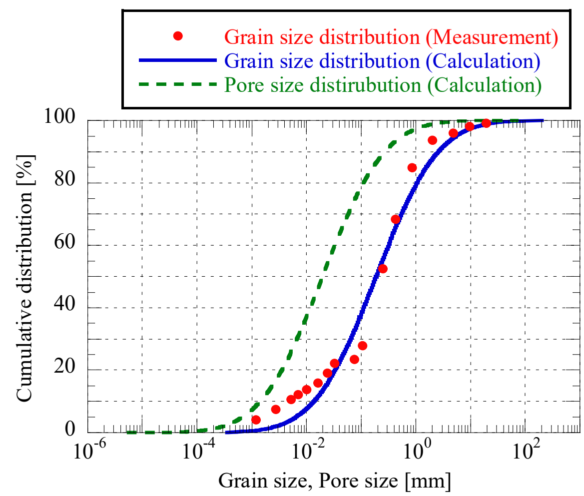



Figure 13 Grain size distribution and pore size distribution

Table 1 List of the assumptions

| | | Assumptions | Remarks |
|---|---|---|---|
| 1 | Grain size distribution | The grain size distribution is assumed to be the logarithmic normal distribution. | The different distribution curves can be accepted when the parameters prescribed the curves should be identified. |
| 2 | Pore size distribution | The pore size distribution is assumed to be the logarithmic normal distribution as well as the grain size distribution. The coefficient of variance is also same as that of grain size distribution. | These assumptions would be expected to be improved with the development of computer sicence and electrical-elctronic technology such as CT (Computer Tomography), Image processing etc.. |
| 3 | Shape of soil particles | The shape of soil particles is assumed to be spherical. | This assumption is similar to one that soil particles are sperical to obtain the diameter of soil particle in sieve analysis and sedimentation analysis, i.e., in the grain size analysis the diameter of soil particle is estimated based on the assumption that the shape of soil particle is spherical. |
| 4 | Distribution of contact angle | The shape of distribution of contact angle is assumed to be pentagonal base on the experimental results by Oda (1972). | The microscopic state of soil would be expected to be estimated quantitatively with the development of computer science and electrical-electronic technology such as CT (Computer Tomography), Image processing etc.. |
| 5 | Distribution of inclination angle in EPM | The distribution of inclination angle in EPM which is the direction of predominant flow in EPM is assumed to be same as that of contact angle. | This assumption would be expected to be improved by the microscopic observation of pore water flow with the development of computer science and electrical-electronic technology such as CT (Computer Tomography), Image processing etc.. |

## 4. CONCLUSIONS

It is concluded as follows.

- The microscopic state of soil particle structure in soil can be estimated by Equations (17) and (20), and the microscopic state of pore structure in soil can be estimated by Equations (21) and (25).
- All of the parameters included in Equations (17), (20), (21) and (25) except $\zeta_c$ can be determined by the densities of soil particle and water, the grain size distribution curve, the void ratio and the water content which are obtained by the laboratory soil tests.
- The development of measuring technologies such as the laser diffraction and image processing in the computer science field are expected to measure the contact angle of particulate material in the future.

The authors proposed the initial part of their microscopic mechanical model in this paper. The whole proposed model will be published in a textbook under CRC Press in the near future.

## 5. ACKNOWLEDGEMENT

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### Appendix 1 Mean value and Variance of Probability Distribution

Let's consider the sample population which includes $N$ samples. When the value for the $i$-th sample with one attribute is expressed as $x_i$, the mean value of random variable $X_i$ can be obtained by the following equation for the discontinuous sample space:

$$\mu = \bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i \tag{A1.1}$$

where $\mu$, $\bar{x}$ : mean value of random variable $X_i$.

The mean value $\bar{x}$ can be expressed as $E[X_i]$ and thus Equation (A1.1) is written as follows:

$$\sum_{i=1}^{N} x_i = N \cdot \bar{x} = N \cdot E[X_i] \tag{A1.2}$$

Introducing the histogram (frequency distribution), the mean value is obtained by the following equation:

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{n} N_i \cdot x_i = \sum_{i=1}^{n} \frac{N_i}{N} \cdot x_i = \sum_{i=1}^{n} p(x_i) \cdot x_i \tag{A1.3}$$

where $N_i$: number of samples with sampled value of $x_i$,
$p(x_i)$: relative frequency (probability mass function).

Then $N_i$ is obtained from Equation (A1.3) as follows:

$$N_i = N \cdot p(x_i) \tag{A1.4}$$

The variance of random variable $X_i$ is obtained from the following equation.

$$\begin{aligned}\sigma^2 = V[X_i] &= \sum_{i=1}^{N} \left(x_i - \bar{x}\right)^2 p(x_i) \\ &= \sum_{i=1}^{N} x_i^2 p(x_i) - 2\bar{x}\sum_{i=1}^{N} x_i p(x_i) + \bar{x}^2 \sum_{i=1}^{N} p(x_i) \\ &= E\left[X_i^2\right] - 2\bar{x}^2 + \bar{x}^2 = E\left[X_i^2\right] - \bar{x}^2\end{aligned} \tag{A1.5}$$

where $\sigma^2$, $V[X_i]$: variance of random variable $X_i$.

Equation (A1.5) is rewritten as follows:

$$V[X_i] = E\left[X_i^2\right] - \mu^2 \tag{A1.6}$$

For the continuous sample space Equations (A1.1) and (A1.4) are expressed as follows.

$$\mu = \bar{x} = E[X] = \int_{-\infty}^{\infty} x \cdot f(x) dx \tag{A1.7}$$

where $f(x)$: probability density function of random variable $X$.

$$N_i = N \cdot f(x) \cdot dx \tag{A1.8}$$

The variance of random variable $X$ is expressed for the continuous sample space, corresponding to Equation. (A5) as follows:

$$\sigma^2 = V[X] = \int_{-\infty}^{\infty} \left(x - \bar{x}\right)^2 f(x) dx \tag{A1.9}$$

For the continuous sample space Equation (A1.6) can be similarly reduced from Equation (A1.9).

In the case that a sample has two attributes designated by $x$ and $y$ with the probability density function $f(x, y)$ and additionally the random variables $X$ and $Y$ are independent, the following relation can be obtained.

$$f(x, y) = f_x(x) \cdot f_y(y) \tag{A1.10}$$

where $f(x, y)$: joint probability density function of random variables $X$ and $Y$,
$f_x(x)$: marginal probability function of random variable $X$,
$f_y(y)$: marginal probability function of random variable $Y$.

Introducing the function of random variables $X$ and $Y$, $\varphi(X, Y)$, the mean value of $\varphi(X, Y)$ is obtained as follows:

$$E[\varphi(X,Y)] = \iint \varphi(x, y) \cdot f(x, y) dxdy \tag{A1.11}$$

When the random variables $X$ and $Y$ are independent, Equation (A1.11) is rewritten as follows:

$$E[\varphi(X,Y)] = \iint \varphi(x, y) \cdot f_x(x) \cdot f_y(y) dxdy \tag{A1.12}$$

### Appendix 2 Number of Soil Particles per Unit Volume

Let's consider that Figure A2.1 is obtained from the grain size analysis using the undisturbed soil block of which the void ratio, the whole volume and dry mass are denoted by $e$, $V$ and $M_s$, respectively.

Then the following equation is obtained from the phase diagram shown in Figure 2.

$$M_s = \frac{V}{1+e}\rho_s \tag{A2.1}$$

When the volume is unity, i.e., $V$=1, Equation (A2.1) is rewritten as follows:

$$M_{s,unit} = \frac{1}{1+e}\rho_s \tag{A2.2}$$

Figure A2.1 shows a cumulative grain size distribution. The vertical axis (percent finer by mass) in Figure A2.1 is divided into $n$ equal intervals (for example $n$=180). The diameter corresponding to the $i$-th interval of vertical axis is denoted by $D_{s,i}$ and the interval is denoted by $\Delta D_{s,i}$ as shown in Fig.A2.1. The mass of soil in the $i$-th interval is denoted by $M_{s,unit,i}$, and then the following equation is obtained.

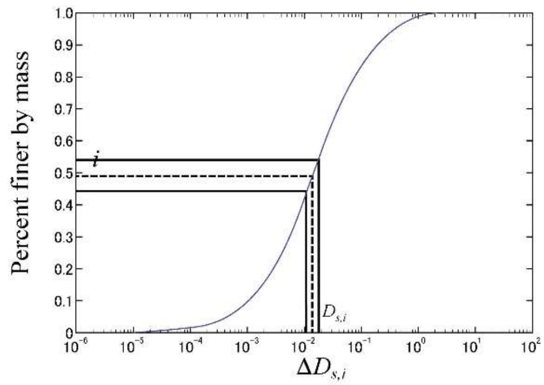



Figure A2.1 Cumulative grain size distribution divided into *n* intervals

$$M_{s,unit} = \sum_{i=1}^{n} M_{s,unit,i} \tag{A2.3}$$

When the number of soil particles included in the *i*-th interval is denoted by $N_{prt,i}$, the following equation is obtained.

$$M_{s,unit,i} = N_{prt,i} \cdot \frac{D_{s,i}^{3}}{6} \pi \cdot \rho_s \tag{A2.4}$$

Applying Equation (A1.8) to Equation (A2.4) and using Equation (21), the following equation is obtained:

$$M_{s,unit,i} = M_{s,unit} \cdot f_s(D_{s,i}) \cdot \Delta D_{s,i} \tag{A2.5}$$

Substituting Equations (A2.2) and (A2.4) into Equation (A2.5), the following equation is obtained:

$$N_{prt,i} \cdot \frac{D_{s,i}^{3}}{6} \pi = \frac{1}{1+e} \cdot f_s(D_{s,i}) \cdot \Delta D_{s,i} \tag{A2.6}$$

Equation (A2.6) is rewritten as follows:

$$N_{prt,i} = \frac{1}{1+e} \cdot \frac{6}{D_{s,i}^{3} \cdot \pi} \cdot f_s(D_{s,i}) \cdot \Delta D_{s,i} \tag{A2.7}$$

As the void ratio and the grain size distribution on the right hand side of Equation (A2.7) are known, $N_{prt,i}$ is determined for a given $D_{s,i}$. Finally the number of soil particles per unit volume $N_{prt}$ can be obtained as follows.

$$N_{prt} = \sum_{i=1}^{n} N_{prt,i} = \frac{1}{1+e} \cdot \frac{6}{\pi} \cdot \sum_{i=1}^{n} \frac{1}{D_{s,i}^{3}} \cdot f_s(D_{s,i}) \cdot \Delta D_{s,i} \tag{A2.8}$$

It is found from Eq. (A2.8) that the number of soil particles per unit volume $N_{prt}$ can numerically calculated by the void ratio and the grain size distribution.

**Appendix 3 Derivation of $D_{cha}$**

Figure A3.1 (a) shows the simple cubic packing of 16 uniform spheres 4 (column: *i*=4)×4 (row: *j*=4) with the density of particles $\rho_s$ and the diameter $D_s$. Here let's consider the force per unit area at the bottom plane. For a column including 4 spheres (i.e., number of rows *j*=4), the applied force of the *i*-th column $F_i$ is obtained as follows:

$$\vec{F}_i = \sum_{j=1}^{4} \rho_s g \cdot \frac{D_{s,j}^{3} \cdot \pi}{6} = 4\rho_s g \cdot \frac{D_s^{3} \cdot \pi}{6} \tag{A3.1}$$

Using Equation (A3.1), the average force acting on the bottom plane is obtained as follows:

$$E[\vec{F}_i] = \frac{1}{4} \sum_{i=1}^{4} \rho_s g \cdot \frac{D_{s,j}^{3} \cdot \pi}{6} = 4\rho_s g \cdot \frac{D_s^{3} \cdot \pi}{6} \tag{A3.2}$$

where $E[\vec{F}_i]$: average force of the *i*-th column (*i*=1~4) acting on the bottom plane,

Thus, the resultant force per unit area at the bottom plane $\vec{F}$ is obtained as follows:

$$\vec{F} = \frac{4\rho_s g \cdot \frac{D_s^{3} \cdot \pi}{6} \times 4}{4 \cdot D_s} \tag{A3.3}$$

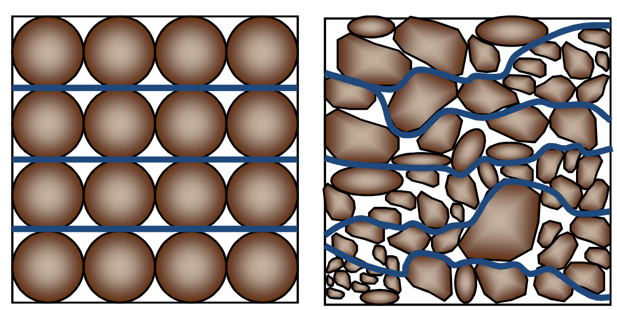
An alternative expression of Equation (A3.3) is as follows.

(Resultant force per unit area)
= (Weight of spheres in single row)×(Number of rows)／(Area) (A3.4)

Figure A3.1(a) shows the contact planes for uniform particles where the contact planes are flat surfaces and Figure A3.1(b) shows random particles in shape and size where the contact plane is irregular and curved. To obtain the resultant force per unit area of bottom plane for irregular soil particles in shape and size, it is necessary to determine the number of rows in Eq. (A3.4) in a similar way as for simple cubic packing. Therefore, a new parameter called characteristic length $D_{cha}$ for soil with irregular particles is introduced. Here let's hypothesize that a cube with unit length contains $N_{prt}$ dry uniform spheres with the diameter $D_{cha}$, where $N_{prt}$ is the number of particles defined by Equation (A2.8). Then the average force per unit area of the bottom plane (i.e., the pressure due to soil block) is obtained as follows:

$$p_{bottom} = \rho_d \cdot g = \rho_s \cdot g \cdot \frac{D_{cha}^{3} \pi}{6} \cdot N_{prt} \tag{A3.5}$$

where $p_{bottom}$: pressure on the bottom plane of cubic,
$\rho_d$: dry density of soil



(a) Simple cubic packing of uniform spheres (b) Random packing of irregular particles

Figure A3.1 Contact planes for uniform particles and random particles in shape and size

Substituting Equations (8) and (A2.8) into Equation (A3.5), the following equation is obtained:

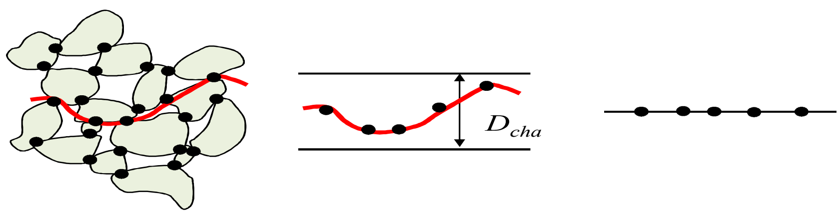
$$\frac{1}{1+e}\rho_s \cdot g = \rho_s \cdot g \cdot \frac{D_{cha}{}^3 \pi}{6} \cdot \frac{1}{1+e} \cdot \sum_{i=1}^{n} \frac{6}{D_{s,i}{}^3 \cdot \pi} \cdot f_s(D_{s,i}) \cdot \Delta D_{s,i} \tag{A3.6}$$

Thus, $D_{cha}$ is obtained as follows:

$$D_{cha} = \left( \sum_{i=1}^{n} \frac{1}{D_{s,i}{}^3} \cdot f_s(D_{s,i}) \cdot \Delta D_{s,i} \right)^{-\frac{1}{3}} \tag{A3.7}$$

where $f_s(D_{s,i})$: probability density function obtained from grain size distribution curve.

Equation (A3.7) shows the characteristic length $D_{cha}$, one of the material constants for a given soil, is determined by the grain size distribution curve as well as the uniformity coefficient and the coefficient of curvature. It may be considered that the characteristic length $D_{cha}$ is the parameter of row thickness in Figure A3.1 to link the body force with the surface force although these forces are definitely separated as independent physical quantities and the thickness of surface is infinitesimal in the continuum mechanics. Figure A3.2(a) shows a curved path connecting through the contact points of particles with irregular size and shape. Figure A3.2(b) shows the thickness of path. Figure A3.2(c) shows the hypothetical flat surface corresponding to planes in Figure A3.1(a).




(a) Curved path connected with contact points

(b) Thickness of layer through path

(c) Flat plane corresponding to curved path

Figure A3.2 Physical meaning of $D_{cha}$