Finite Difference Analysis of Raft Foundations under Vertically Static Loads

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ABSTRACT: This paper introduces the Finite Difference formulas derived from the thin-plate theory on raft foundation with the influences from boundary conditions. The proposed solution was established for a raft located at the ground surface, where the bending moments and shears along the edges of the foundation were presumed vanished. The solutions were examined with the three-dimensional Finite Element analysis on numerical foundation subjected to uniformly load. Computer program WERAFT-S was developed for raft foundation on elastic soils under vertically static load. Effectiveness of a couple of soil spring models was discussed with the validation of WERAFT-S to three dimensional Finite Element solutions.
Keywords: Finite difference analysis, raft foundation, plate theory, boundary conditions, soil springs

1. INTRODUCTION

The load-displacement relations of raft foundations can be modeled by two-dimensional or three-dimensional analyses. Both modeling can be done by the Finite Element Analysis (FEA). For two-dimensional analysis, the foundation can be treated as a one-dimensional beam on a set of soil springs. The analysis is termed as Beam on Elastic Foundation or Winkler foundation where the soil springs can be elastic or inelastic. This type of solution is applicable when the length-to-width ratio (L/W) of the foundation (where L is the length, W is the width) exceeds 10. The two dimensional analysis has been discussed for decades (Biot 1937; Mathews 1958; Bowles 1977; Ting and Mockry 1984; Jones 1997; Chen 1998; Tomlinson and Boorman 2001; Dinev 2012; Chiou et al. 2016; Chang et al. 2016). Among the available solutions, the Finite-Difference Analysis (FDA) of the foundation can be found by Bowles (1977), Tomlinson and Boorman (2001) on settlements of the foundation under static loads. Chang et al. (2016) on the other hand has demonstrated that the one-dimensional beam on soil-pile elements can be used to simulate the seismic behaviors of the foundation subjected to horizontal ground motions. A recent report comparing various solutions of the beam analysis can be found in Omer and Arbabi (2015).

The three-dimensional analysis can be conducted by treating the raft foundation as a two-dimensional plate (or mat) resting upon the ground soils. Complexities and versatility are involved in deriving the governing equations of foundation's loading behaviors. Analytical solutions in this regard have been presented (Timoshenko and Krieger 1959; Vlasov and Leontev 1966; Kukreti and Ko 1992). However owing to the complexities involved in the analytical expressions, these solutions were rarely used in engineering practice. A numerical solution based on a series of strip footings was then suggested by Poulos (1991). This solution was later improved by Poulos (1994) with the approximate numerical solutions for plate on soil continuums from boundary integrals. At the present time, modeling of the 2D raft can be found in many research studies on piled raft foundation (Randolph 1983; Clancy and Randolph 1993; Horikoshi and Randolph 1996; Kobayashi et al. 2009; Kitiyodom and Matsumoto 2002; Kitiyodom et al. 2005). In general, discrete foundation can be established using two approaches, 1. Equation of Motion, and 2. Wave Equation. The former solves the foundation displacements based on stiffness matrix (or via the mass/damping matrices for dynamic problems) of the foundation. Raft was modeled using two-dimensional finite elements and/or simplified grid (beam-column) elements underlain by a set of soil springs.

The latter requires a governing differential equation of the deformational characteristics of the foundation. The representative solution of such approach can be found in Bowles (1977) where the two-dimensional Finite Difference Analysis was introduced to solve the problem. Comprehensive summaries regarding the design and analysis of the raft foundations can be found in Gupta (1997) and Hemsley (1998). The FD analysis needs further attentions to take into account of the effects of boundary conditions when the foundation is relatively large and flexibility of raft became significant. In contrast with the one suggested by Bowles (1977), FDA based on FLAC program (Itasca 2017) was established in examining the stress continuities of the material particles. Applying them to the nodes of the foundation requires considerable amount of iterations to ensure the equilibriums.

This study proposed a three-dimensional FDA for a surface raft foundation subjected to vertically static loads with boundary effects. The governing equations were adopted from the thin plate theory. Computer program WERAFT-S was developed and verified with three-dimensional FEM analysis.

2. METHODOLOGIES

Theory of Plate can be categorized for thin plate and thick plate. In general if the thickness of the plate (D) is less than a tenth t of the width (W) of plate, it can be regarded as thin-plate. The Kirchhoff-Love classical plate theory was suggested on thin plates, whereas the Mindlin-Reissner plate theory is applicable to the thick plates. The thick plate theory considers the in-plane shear strains whereas the thin plate theory does not.

2.1 Governing Equation

According to Timoshenko and Woinowsky-Krieger (1959), governing equation of the vertical displacements of a thin plate subjected to vertically uniform load (q) and point load (P) can be written as follows,

$$\frac{\partial^4 w}{\partial x^4}+\frac{2\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}=\frac{12q(1-\nu^2)}{ED^3}+\frac{12P(1-\nu^2)}{ED^3(\partial x\partial y)} \quad (1)$$

where w is the vertical displacement of the raft, ν and E are the Poisson's ratio and Young's Modulus of raft, *D* is thickness of the raft, and *x* and *y* are the spatial variables.

2.2 Boundary Conditions

For a raft foundation located at the ground surface as shown in Fig. 1, the moments and shear forces are assumed vanished at edge of the foundation. By looking at the foundation with plan view, the top and bottom edges of the raft where *y*=constant, M_x (bending moment rotating at the *x*-direction) and V_y (vertical shear force at the surface normal to the *y*-direction) can be written as follows,

$$Mx=-B(\partial y^2+\nu\partial^2 w/\partial x^2)=0 \quad (2)$$

$$Vy=-B\left[\frac{\partial^3 w}{\partial y^3}+\frac{(2-\nu)\partial^3 w}{\partial y\partial x^2}\right]=0 \quad (3)$$

where B is the expression of $ED^3/(12(1-\nu^2))$. Similarly, at the left and right edges of the raft where *x*=constant, the boundary conditions M_y and V_x are:

$$My = -B(\partial^2 w/\partial x^2 + v\partial^2 w/\partial y^2) = 0 \quad (4)$$

$$Vx = -B\left[\frac{\partial^3 w}{\partial x^3} + \frac{(2-v)\partial^3 w}{\partial x \partial y^2}\right] = 0 \quad (5)$$

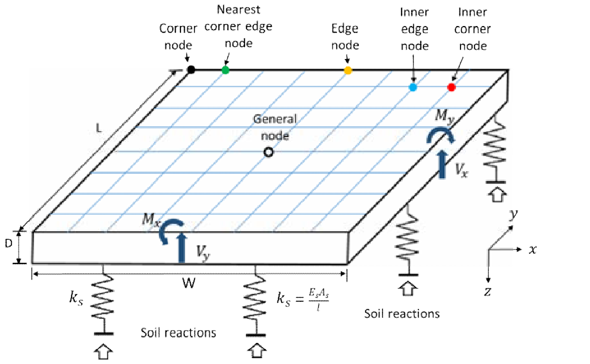

Figure 1 Layout and discrete nodes of a vertically loaded raft foundation

2.3 Soil Springs

For governing equation of the raft foundation with the influences of soil reactions, a number of models can be used. For simplicity, the compressional spring model was adopted in this study, in which the soil spring constant K_s is simply calculated by E_sA_s/l where E_s is the Young's Modulus of the soil, A_s is the effective area of the soils underneath the nodes of the raft, and l is the length of the soil spring (or effective thickness of the soils). For a surface foundation, the influences of the soil reactions on the raft can be taken into account assuming that their influences are similar to the superstructure loads. Therefore in Eq. (1), q can be replaced by q^* where $q^*=q-\Sigma K_s w_k/A_r = q-(E_s/l)\Sigma A_{sk} w_k/A_r$; w_k is foundation settlement at the k^{th} node, A_{sk} is the area of soil spring under the k^{th} node, and A_r is the total area of the raft which equals to ΣA_{rk}, where A_{rk} stands for the area of raft at the k^{th} node. Now defining q_k^* as the modified load allocated at the k^{th} node, for simplicity, q_k^* can be approximated as $q-(E_s/l)w_k(A_{sk}/A_{rk})$, where (A_{sk}/A_{rk}) is called as the area ratio at the k^{th} node.

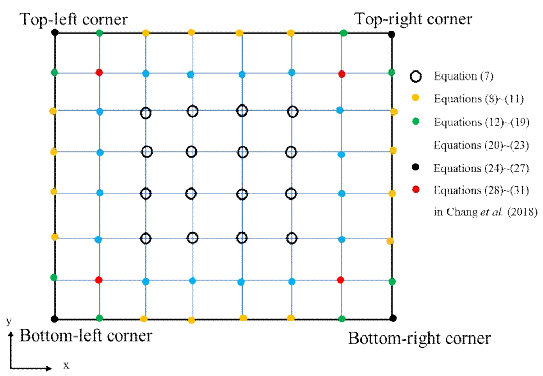

Figure 2 Allocations of the corresponding equations of the proposed analysis

This assumption matches closely with the flexible foundations where the raft size is relatively large. For smaller raft foundation that behaves more rigidly, this assumption should be not applicable. One can adopt other types of soil spring model. For example, the Lysmer Analog soil spring can also be used. The total soil spring constant of the whole raft foundation could be calculated as 4Gsro/(1νs), where Gs is the shear modulus of the soil, νs is the Poisson's ratio of the soil, and ro is the equivalent radius of the raft foundation.

2.4 Finite Difference Formulas

Using the central-difference formulas, the resulting formulations for the nodes at a surface foundation can be derived. The orientation and categories of these nodal points are shown in Fig. 2. Details of the derivations can be found in Lien (2018). Note that the spacing distance between the nodes in *x*- and *y*- directions are kept the same (*i.e.*, $\Delta x=\Delta y=s$) for simplicity of the expressions. Moreover, the point load P applied at arbitrary nodes of the raft can be taken as an extra uniform load applied to that node within the area which is equal to $\Delta x \Delta y$. Fig. 3 shows the nodal points used and the fictitious points encountered in the derivations.

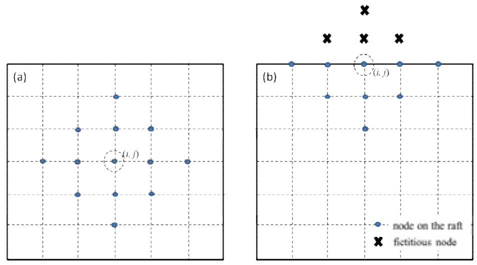

Figure 3 Nodes for derivations of the FD formulas, (a) General node (b) Edge node

With the discrete equations derived, one can easily establish a set of dependent equations for the nodes allocated at the raft. Matrix analysis is required to solve for the foundation displacements. When calculating the soil reactions at the nodes along the edges, the area ratio (A_{sk}/A_{rk}) of the soil springs can be represented by a value, *n* defined by dividing the length of the area with the standard width (which is equal to 1.0 m) of the soils underneath the general nodal points. For the nodes at the corners, the area ratio of the soil spring would be expanded to $n \times n/1.0$ as n^2. The spacing distance between two adjacent nodes is kept as 1 m for simplicity. Fig. 4 depicts the area of soil springs for nodes at the edges and the corners. As a result, a computer program WERAFT-S was suggested (Lien 2018).

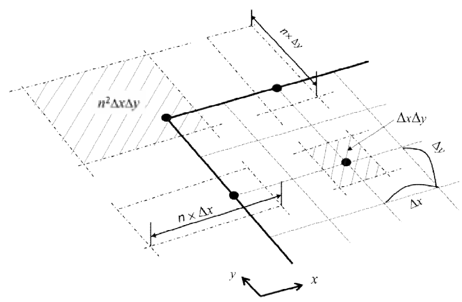

Figure 4 Schematic layout of the area ratio (A_s/A_r) in calculating soil reactions at the nodes

3. VALIDATION

Validation of the proposed analysis is presented next. Assuming that a square raft has width (W) of 26 m and thickness (D) of 1 m. Young's modulus (E), Poisson's ratio (ν), and unit weight (γ) of the concrete raft foundation are 3×10^4 MPa, 0.15, and 24 kN/m^3, respectively The soils underneath the foundation are assumed to have a shear wave velocity (V_s) of 150 m/s, Poisson's ratio (ν_s) of 0.4, and unit weight (γ_s) of 19 kN/m^3. Note that $V_s^2\times\gamma_s/g=G_s$, where g is acceleration of the gravity, G_s is Shear modulus of the soil. Note that $E_s=2G_s(1+\nu_s)$, therefore E_s can be calculated. The uniform load applied on top of the foundation is 100 kPa. Three dimensional FEA solutions of the numerical model were first obtained from Midas-GTS (Midas 2017) varying the dimensions of the FE zone. Essential boundaries (*i.e.*, roller and hinge) were adopted in the modeling.

Eight-node brick elements were mainly used to generate the structural system.

Fig. 5 shows that the foundation settlements obtained from FE analysis will gradually converge by increasing the FE zone at 60~200 m (note that the length, width and thickness of the zone were kept the same for the analytical zone). The settlements became stable when the analytical zone was increased to 200 m x 200 m x 200 m. Therefore the FEA of the study was conducted using the optimized core. It can be seen that the foundation settlements are different at various locations. In general, the center exerts the largest settlement whereas the corner has the smallest settlement. The soil mass involved in the FE analysis seems to have significant influence on foundation settlements. Validation of the WERAFT-S analysis was then conducted to compare with the FE solutions. The influences of the area ratio ($A_{sk}/A_{rk}=n$) of the edge node, the shear wave velocity (V_s) and the Poisson's ratio (v_s) of soil, the thickness of raft (D), and the load intensity (q) were studied to see the appropriateness of the length (l) of soil spring.

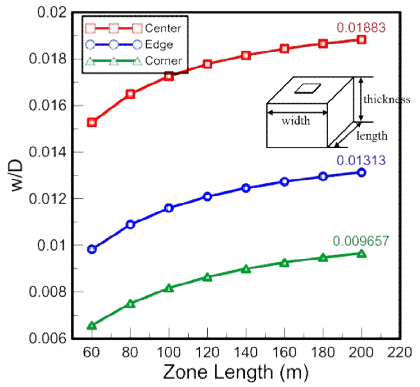

Figure 5 Influence of FE analytical zone on the foundation settlements

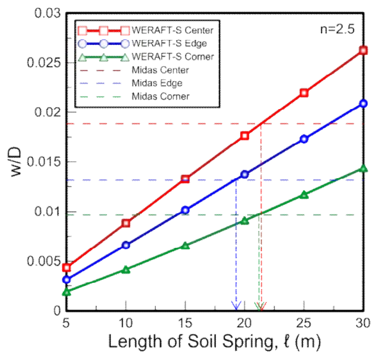

Figure 6 Effect of area ratio at edge nodes on foundation settlements

Fig. 6 shows the effects of area ratio (n) at the edge nodes and length of the soil springs (l) from the WERAFT-S analysis on normalized settlements (w/D) appeared at center, edge, and corner of the foundation. Note that Δx and Δy are kept as one meter in this study. Therefore the raft will have 729 (27×27) nodal points. The horizontal dash lines represent the settlements from FE analysis. Note that when n=2.5 and the length of soil spring (l) approximates 21 m seems to provide compatible solutions with the FE analysis. The matches were found very sensitive for the settlement occurred at the corner. As the area ratio increases from 2 to 3, the appropriate length of the soil spring for the corner settlement is increased rapidly from 17 m to 25 m, where the corresponding ones at the edge and the centre are varying insignificantly around 18~21 m and 21 m, respectively.

4. EFFECTS OF SOIL SPRINGS

The influences of effective thickness of the soil spring were studied. It was found that the shear wave velocity of the soil is the most important parameter if the compressional spring was adopted. Fig. 7 depicted the results. More details of the discussions can be found in Chang *et al.* (2018). Fig. 8 shows the matching of the displacements at centre of the foundation with the FEM ones on numerical model varying the shear wave velocity of soil. Corresponding interpretation of the appropriate thickness of the soil springs with respect to shear wave velocity of the soil can be obtained. Note that when V_s is at 120~180 m/s, the effective thickness of the soil spring will be around 32~17 m. The effective thickness of the soil spring seems to be much smaller than what was expected (for square foundation, effective depth of the foundation is around 2B). The reason is because that the force acting along the spring is the same while the real stress magnitudes of the foundation will gradually decrease with the depth.

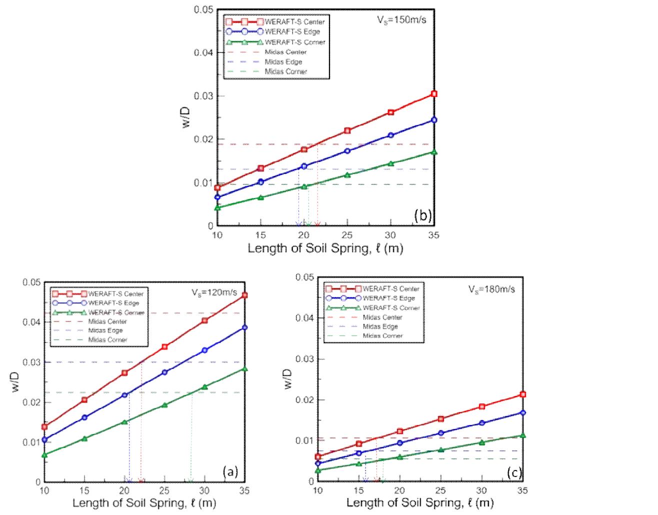

Figure 7 The effects of shear wave velocity of soil and length of underneath soil spring on foundation settlements, (a) V_s = 120m/s (b) V_s = 150m/s (c) V_s = 180m/s

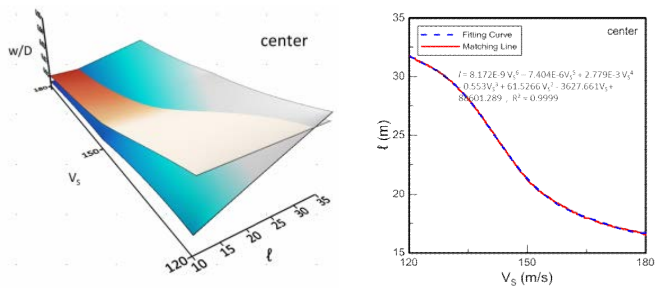

Figure 8 Matching and interpretation of the optimized effective thickness of soil springs underneath the raft foundation

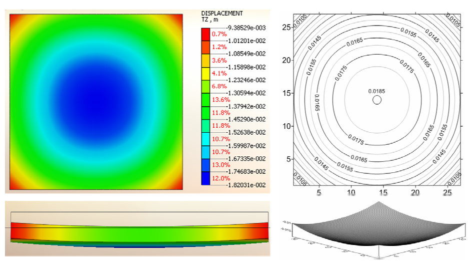

Figure 9 Comparisons of displacement contours from Midas-GTS analysis and WERAFT-S analysis

Comparisons of the displacement contours of the standard model foundation from both WERAFT-S and FEM can be found in Fig. 9. To show the possible influences of the soil springs using Lysmer Analog model, the results were listed in Table 1. It can be found that the deviations between the WERAFT-S and FEM are smaller when finding the optimized solutions of simplified soils springs. The predictions using Lysmer Analog springs need to further examinations.

Table 1 Comparisons on Foundation Displacements Obtained from FEA and WERAFT-S using Different Soil Spring Models

Shear wave velocity (m/s)	**Analysis**	**Midas**	**w/ EA/*l* model**	**w/ Lysmer Analog model**
	location	**foundation displacement (mm)**		
120	center	42.3	40.3	28.7
	edge	30.0	31.8	21.7
	corner	22.5	22.7	13.2
150	center	18.8	18.6	18.6
	edge	13.1	13.7	13.6
	corner	9.7	9.2	7.9
180	center	10.6	11.1	13.0
	edge	7.5	7.9	9.2
	corner	5.6	5.1	5.2

5. CONCLUDING REMARKS

This paper presents a newly proposed finite difference analysis WERAFT-S for the surface raft foundation under vertical loads using the Thin-Plate Theory. The effects of boundary conditions were considered to derive the formulas of the raft foundation settlements. Elastic compressional soil springs were used to simulate the soil resistances underneath the raft. Solutions of the proposed analysis were verified on a numerical raft foundation located at the surface of a ground consisting of elastic soils (where V_s=150 m/s and ν_s=0.4). Vertical uniform load of 100 kPa was mounted on the foundation. Validation of the analysis was made with those from three-dimensional finite element analysis (FEA) using Midas-GTS. The effective thickness of the compressible soil springs underneath the raft was studied. It was found that the effective thickness of the soil at 17~32 m (for Vs at 180~120 m/s) can provide agreeable solutions with the FEA. Although the compressible soil spring model has some defects in the real applications, it has been shown that the adequate solutions can be still achieved. On the other hand, the Lysmer Analog springs seem to provide much smaller foundation displacements on stiffer site. The results need further studies with the concerns of uneven soil stiffness distributed underneath the foundation. If the raft foundation is rigid enough where the boundary effects can be neglected, then the solution proposed by Bowles (1977) can be used assuming uniform settlement. Vertical loads such as uniform, non-uniform, and point loads need to be taken into account with pre-calculations.

6. ACKNOWLEDGEMENT

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