GENERALIZATION OF THREE-MODULI INCREMENTAL NON-LINEAR CONSTITUTIVE MODELS FOR SOILS

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ABSTRACT

In this paper, two three-moduli incremental non-linear constitutive models are presented for triaxial stress condition. The method for the determination of three moduli (K,G,J) in one of the incremental non-linear constitutive models using conventional triaxial test data is discussed. Mathematical functions are suggested for best data fittings. The three-moduli models can consider shear-compression (or shear-dilation) by introducing the coupling modulus J (or $\bar{J}$). The three moduli K,G,J are determined using data from an isotropic consolidation test and three CID triaxial shear tests. The computed results using the calibrated *KGJ* model are compared with the measured test data from three drained triaxial shear tests in constant p'. The main focus of the paper is on (a) a new approach in the generalization of the two three-moduli models for triaxial stress condition and (b) the interpretation of physical meanings of the two generalized models. The two new generalized forms (models) are derived based on incremental isotropic hypoelasticity theory. Both the tensor form and the matrix form of the two new generalized stress-strain relationships are presented. Physical meanings of the generalized constitutive relationships are discussed with an especial attention to the items associated with the coupling modulus J (or $\bar{J}$).

INTRODUCTION

It is commonly recognized that the constitutive model for the stress-strain-strength behavior of soils is the key to meaningful and accurate prediction of the performance of geotechnical soil structures (Chen and Mizuno, 1990; Yin, 1990; Chen, 1994). The research in constitutive modeling for soils has been a very active area with a long history (Chen and Mizuno, 1990). The stress-strain behavior of soils under static loading is non-linear, plastic, time-dependent. This paper will study mainly the non-linear and plastic stress-strain behavior of sandy soils, the time-dependency (or strain-effects) of which may be ignored.

To simulate the non-linear and plastic behavior of soils, a number of elastic-plastic models (Chen and Mizuno, 1990) have been proposed from Drucker's stability postulate (Drucker, 1951), for example, Drucker and Prager's model (1952) and the Modified Cam-Clay model (Roscoe and Burland, 1968). As a different approach, hypoelasticity (Truesdell, 1955) has been used to model the non-linear and plastic stress-strain behavior of soils. One of the most popular hypoelastic models for soils is the two-moduli (using stress-dependent E and ν or E and K) incremental stress-strain model proposed by Duncan and Chang (1970). The determination of the two moduli E and ν or E and K can be found in Duncan *et al.* (1980) and Duncan (1981). Domaschuck and Villiappan (1975) suggested a two-moduli model using stress-dependent shear modulus G and bulk modulus K. One of the main limitations is that the two models cannot consider shear stress induced compression/expansion and mean-stress induced shear strains, which are common for most soils subject to large loading. Darve *et al.* (1986) suggested models with more than three moduli. However, the determination of those moduli requires special tests and is often difficult. To consider shear-dilation behavior of soils, Yin and Yuan (1985) suggested a three-moduli model. But this model cannot consider mean-stress induced shear strain. Yin *et al.* (1989, 1990) developed an improved version of Yin and Yuan's model (1985) to consider both shear stress induced volume strain and mean stress induced shear strain using moduli K,G,J. This three-moduli model is only for triaxial stress states.

Darve (1990) considered the incremental type of constitutive relations is quite a general type of constitutive models including endochronic models. In fact, elastic-plastic models, such as Drucker and Prager's model and the Modified Cam-Clay model are of the incremental form. The non-linear models modified from hypoelasticity are also of the incremental form. All these models are similar with the same incremental form but different in details in establishing these models and probably different in some predicting features.

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Some people think the hypoelastic models are out of date. It is noted that Duncan and Chang's two-moduli model is implemented in a commercial finite element program- SIGMA/W (Geo-Slope 1998) and is still used by geotechnical engineers. Recently, Yang *et al.* (1998) used Duncan and Chang's model in finite element analysis of deep excavations. As long as the hypoelastic models are still used in geotechnical engineering analyses, it is necessary to further improve these models, for example to overcome those limitations mentioned early. There is a misunderstanding on hypoelastic models. Some people consider that hypoelastic models are non-linear elastic models. Therefore, these models cannot be used to describe the irreversible or elastic-plastic stress-strain behavior of soils. This opinion is incorrect. Based on the definition of hypoelasticity, infinitesimal incremental strains are reversible only under loading of infinitesimal incremental stresses. The total strains may be irreversible. However, the feature of irreversibility of hypoelasticity may not be suitable for describing the irreversible stress-strain behavior of soils. Thus, it is common that different moduli are used for unloading/reloading behavior of soils with a loading-unloading/reloading criterion. The practice of using different moduli in loading and unloading/reloading is similar to that in elastic-plastic modeling. Truesdell (1955) and Coon and Evans (1971) pointed out that the incremental form of hypoelasticity has no difference from the form of incremental elastic-plastic models except for details of moduli or terms in the constitutive matrixes.

With the background information above, this paper presents new development in modeling the stress-strain behavior of soils using three moduli based on hypoelasticity with modification for unloading and reloading. A different form using three moduli $\overline{K},\overline{G},\overline{J}$ based on the model proposed by Yin *et al.* (1989, 1990) is proposed. This paper suggests a new method for the determination of moduli K,G,J. Two new generalized forms based on the two forms of the three-moduli models (using K,G,J or $\overline{K},\overline{G},\overline{J}$) are derived for a general stress state based on incremental isotropic hypoelasticity.

THREE MODULUS NON-LINEAR CONSTITUTIVE MODEL

Yin *et al.* (1989, 1990) suggested a three-moduli model in the form

$$\begin{Bmatrix} d\varepsilon_v = \frac{1}{K}dp' + \frac{1}{J}dq \\ d\varepsilon_s = \frac{1}{J}dp' + \frac{1}{3G}dq \end{Bmatrix} \tag{1}$$

where in triaxial stress condition, the volumetric strain $\varepsilon_v = \varepsilon_1 + 2\varepsilon_3$, the generalized shear strain $\varepsilon_s = \frac{2}{3}(\varepsilon_1 - \varepsilon_3)$, effective mean stress $p' = \frac{1}{3}(\sigma_1' + 2\sigma_3')$, and the deviator stress $q = \sigma_1 - \sigma_3$. Here, compressive stresses and compressive strains are positive. In (1), K is the bulk modulus which represents the volumetric stiffness of the soil with respect to the effective mean stress increment dp' and K shall be positive. The G in Eq. (1) is the shear modulus which controls the shear strain with respect to dq. The G may be positive for strain hardening and negative for strain softening. The J in (1) is the coupling modulus which accounts for the volumetric strain produced by an increment dq and also the shear strain produced by an increment dp'.

The formulation in Eq. (1) assumes that the dp'-$d\varepsilon_s$ coupling and the dq-$d\varepsilon_v$ coupling are controlled by the same J-modulus. Positive dilation, that is, expansion during shearing, is associated with $J < 0$. The compression during shearing is associated to $J > 0$. If there is no dilation or no induced anisotropy, the coupling modulus $J = \infty$.

The three moduli K,G,J are considered here for describing the stress-strain behavior of soils in loading with non-linear and plastic deformation. For unloading and reloading, the same formulation as in Eq. (1) is assumed, but with three different moduli denoted as K^e,G^e,J^e.

Equation (1) can be inversed as follows:

$$\begin{Bmatrix} dp' = \overline{K}d\varepsilon_v - \overline{J}d\varepsilon_s \\ dq = -\overline{J}d\varepsilon_v + 3\overline{G}d\varepsilon_s \end{Bmatrix} \tag{2}$$

where the moduli $\overline{K},\overline{G},\overline{J}$ are related to K,G,J in Eq. (1) and *vice versa* as:

$$\begin{Bmatrix} \overline{K} = K\frac{J^2}{J^2 - 3KG}, \overline{G} = G\frac{J^2}{J^2 - 3KG}, \overline{J} = \frac{3KGJ}{J^2 - 3KG} \\ K = \frac{3\overline{K}\overline{G} - \overline{J}^2}{3\overline{G}}, G = \frac{3\overline{K}\overline{G} - \overline{J}^2}{3\overline{K}}, J = \frac{3\overline{K}\overline{G} - \overline{J}^2}{\overline{J}} \end{Bmatrix} \tag{3}$$

It is seen from Eq. (3) that if J=∞ (no shear-dilation or shear-compression), then $\overline{K} = K, \overline{G} = G, \overline{J} = 0$. The moduli K,G,J may be a non-linear function of stresses and strains. For unloading/reloading, we have similar relationships between $\overline{K}^e,\overline{G}^e,\overline{J}^e$ and K^e,G^e,J^e.

Conventional triaxial tests can provide three independent curves regarding the stress-strain relationship of soils. These three curves may be used for the calibration of the three moduli K,G,J. The method for the determination of the three moduli are discussed below.

<u>Modulus K:</u> The data from one conventional isotropic compression (consolidation) test may be used for the determination of the bulk modulus K. Isotropic consolidation test data provide the relationship between effective mean stress p' and volumetric strain ε_v. Fitting an appropriate mathematical function f_1 to these data produces an equation:

$$\varepsilon_v = f_1(p') \tag{4}$$

In isotropic consolidation test, the deviator stress q is zero, thus from the first equation in Eq. (1), we have:

$$K = \frac{dp'}{d\varepsilon_v} \quad or \quad \frac{1}{K} = \frac{d\varepsilon_v}{dp'} = \frac{df_1(p')}{dp'} \tag{5}$$

<u>Coupling Modulus J:</u> In isotropically consolidated drained (CID) triaxial shear tests, two independent relationships can be obtained, namely volumetric strain vs. deviator stress $\varepsilon_v - q$, that is, $\varepsilon_v = f_2(q)$ and deviator stress vs. shear strain $q - \varepsilon_s$, that is, $q = f_3(\varepsilon_s)$. The first relationship can be used to determine the coupling modulus J.

In a CID test, the confining stress σ_3' is kept constant, the ratio $\eta = dq/dp' = 3$. If we define the slope J_s as:

$$J_s = \frac{dq}{d\varepsilon_v} = \frac{1}{d\varepsilon_v/dq} = \frac{1}{df_2(q)/dq} \tag{6}$$

From the first equation in Eq. (1), we have:

$$\frac{d\varepsilon_v}{dq} = \frac{1}{K}\frac{dp'}{dq} + \frac{1}{J} \tag{7}$$

Using Eq. (6) and noting $dq/dp' = \eta = 3$, we have:

$$J = \frac{\eta K J_s}{\eta K - J_s} \tag{8}$$

In Eq. (8), if J_s=∞, then $J = -\eta K$. If $\eta K - J_s = 0$, then J= ∞, that is, no shear-dilation or shear-compression.

Shear Modulus G: The relationship of deviator stress vs. shear strain ($q-\varepsilon_s$), that is, $q=f_3(\varepsilon_s)$ can be used to determine the modulus G. The slope ($3G_s$) of the curve $q=f_3(\varepsilon_s)$ may be defined as:

$$G_s=\frac{1}{3}\frac{dq}{d\varepsilon_s}=\frac{1}{3}\frac{df_3(\varepsilon_s)}{d\varepsilon_s} \tag{9}$$

From the second equation in Eq. (1), we have

$$\frac{d\varepsilon_s}{dq}=\frac{1}{J}\frac{dp'}{dq}+\frac{1}{3G} \tag{10}$$

Using Eqs. (9) and (10), the shear modulus G is

$$G=\frac{\eta G_s J}{\eta J-3G_s} \tag{11}$$

In (11), if $J=\infty$, then $G=G_s$. If $\eta J-3G_s=0$, then $G=\infty$ which means the soil is rigid against shearing deformation. The determination of K^e, G^e, J^e using unloading and reloading data is similar to that for K,G,J.

CALIBRATION AND PREDICTION

A sand of fine to medium size was tested for model calibration and verification. The maximum particle size of the sand was 0.8mm, minimum size was 0.06mm. Other physical parameters are d_{50} = 0.28mm, d_{60}/d_{10} = 2.13, the specific gravity G_s = 2.68, e_{max} = 0.42, e_{min} = 0.31, D_r = 57% (medium dense), and r_d= 15.4kN/m^3 (dry unit weight).

The measured data of effective mean stress and volumetric strain from an isotropic consolidation test on the sand are shown in Fig. 1. The data are fitted by a hyperbolic function

$$\varepsilon_v=\frac{p'_{iso}}{c_1+c_2p'_{iso}} \tag{12}$$

The fitted curve is shown in Fig. 1. From the curve fitting, the two constants in Eq. (12) are determined as c_1 = 6867kPa, and c_2= 39.4. Differentiating Eq. (12) and using Eq. (5), the K is

$$K=\frac{(c_1+c_2p'_{iso})^2}{c_1}=\frac{(6867+39.4p')^2}{6867} \tag{13}$$

In Eq. (13), the effective mean stress p'_{iso} under isotropic loading condition is replaced by p' for any monotonic loading. This replacement assumes that the bulk modulus K from isotropic compression tests is valid for other stress paths in monotonic loading. The same assumption was used by Domaschuk and Villiappan (1975).

Three drained CID tests were done with three confining pressures of σ'_3= 98kPa, 196kPa and 392kPa. The measured relationships of ε_v vs. q and q vs. ε_s were normalized by making q as $q^*=q/(\sigma'_3)^m$, where m = 0.85, determined by trial-error. The normalized data were shown in Figs. 2 and 3. The ε_v here is the increase in volumetric strain after isotropic compression.

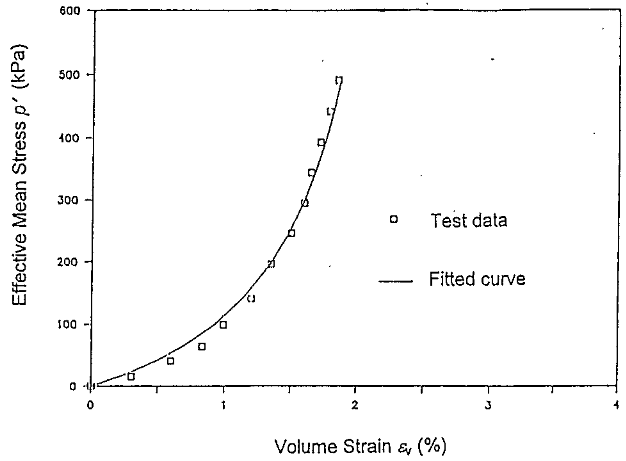


Fig. 1 Measured Data and Fitted Curve of p' vs ε_v in an Isotropic Consolidation Test

The following function provides a good curve fitting of the normalized data in Fig. 2.

$$\varepsilon_v=c_3\frac{q^*(q^*-c_4)}{q^*-c_5} \tag{14}$$

In Eq. (14) the three constants c_3, c_4 and c_5 are determined by the best-fitting. It is found that c_3= 0.0015(kPa)$^{m-1}$, c_4= 7.0(kPa)$^{m-1}$ and c_5= 10.0(kPa)$^{m-1}$. It is seen from Eq. (14) that when $q^*=c_4$= 7.0(kPa)$^{m-1}$, ε_v= 0. It is noted that $q^*<c_5$= 10.0(kPa)$^{m-1}$ for the sand. Other mathematical functions may be used to fit test data. The slope J_s defined in Eq. (6) can be obtained by differentiating Eq. (14)

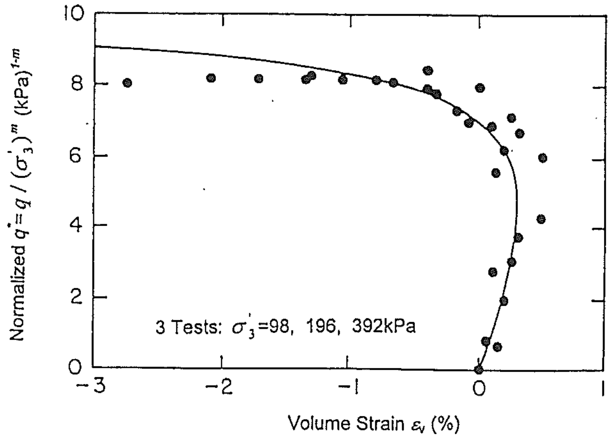


Fig. 2 Measured Data and Fitted Curve of $q/(\sigma'_3)^m$ vs ε_v in CID Tests (Yin et al., 1989)

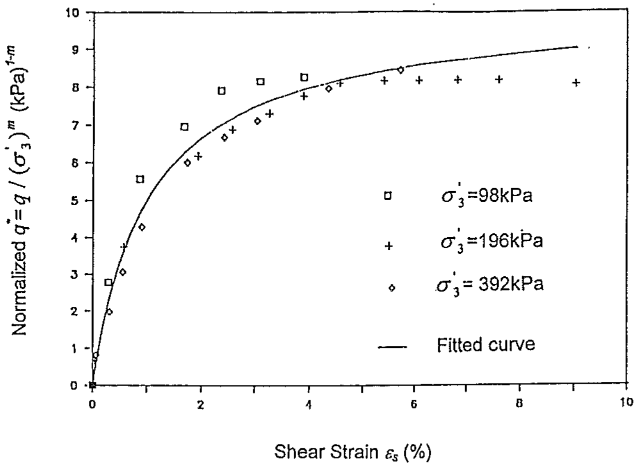


Fig. 3 Measured Data and Fitted Curve of $q/(\sigma_3')^m$ vs ε_s in CID Tests

$$J_s = \frac{dq}{d\varepsilon_v} = \frac{(\sigma_3')^m}{c_3} \frac{(q^* - c_5)^2}{(q^*)^2 - 2q^* c_5 + c_4 c_5}$$

In Eq. (15) when $(q^*)^2 - 2q^*c_5 + c_4c_5 = 0$, $J_s = \infty$. The corresponding q^* is

$$q^* = c_5 - \sqrt{(c_5)^2 - c_4 c_5} = 4.523 \quad (\text{kPa})^{1-m}$$

A hyperbolic function is used to fit the normalized data in Fig. 3

$$q^* = \frac{\varepsilon_s}{c_6 + c_7 \varepsilon_s} \tag{16}$$

By best curve fitting, the two constants in Eq. (16) are determined to be c_6=0.001(kPa)$^{m-1}$, c_7=0.1 (kPa)$^{m-1}$. The fitted curve is shown in Fig. 3. According to Eq. (9), differentiating Eq. (16) leads to

$$G_s = \frac{1}{3}\frac{dq}{d\varepsilon_s} = \frac{(\sigma_3')^m}{3c_6}(1 - c_7 q^*)^2 \tag{17}$$

Having known K, J_s and G_s, the coupling modulus J and shear modulus G can be obtained using Eqs. (8) and (11).

Normalizing q by $(\sigma_3')^m$ reflects the curved strength envelope of the sand. From Eq. (16) the ultimate q^* is equal to $1/c_7 = q_{ult} (\sigma_3')^m = q^*_{ult}$ for $\varepsilon_s = \infty$. As suggested by Duncan and Chang (1970), the deviator stress, q_f, identified to be failure at a limited strain in order to best fit test data is $q_f = Rq_{ult}$ and

$$q_f^* = q_f / (\sigma_3')^m = Rq_{ult} / (\sigma_3')^m = Rq^*_{ult} = \frac{R}{c_7} \tag{18}$$

From Eq. (18), at failure

$$\sigma_3' = (\frac{c_7}{R} q_f)^{1/m} \tag{19}$$

The ratio R may be in the range $0.6 < R < 1$. We know in CID tests (σ_3'=constant), at failure

$$p_f' = \sigma_3' + \tfrac{1}{3} q_f = (\frac{c_7}{R} q_f)^{1/m} + \tfrac{1}{3} q_f \tag{20}$$

Figure 4 shows the fitted curve to a non-linear strength envelope for R = 0.95 and c_7 = 0.1(kPa)$^{m-1}$. If the strength envelope is a straight line, then m = 1.

Using the determined three moduli K,G,J, Eq. (1) can be used to simulate (predict) the stress-strain curves of the sand under different monotonic loading. Isotropically consolidated and constant-p' triaxial shear tests were done on the same sand. Here the measured data are used to check the validation of the KGJ model, noting that the constant-p' test data have not been used for the determination of the three KGJ moduli. Figure 5 shows the measured and predicted curves of q vs ε_s and ε_v vs ε_s. It is seen that, overall, the KGJ model gives a reasonable prediction of the stress-strain responses of the sand. It is noted that for the test with p'= 392 kPa, the predicted values are slightly larger than the measured value. The reason is that the strength envelope of the sand is normally not a straight line, but curved. In other words, the friction angle normally decreases with increasing normal stress. The proposed KGJ model has tried to deal with the curved strength envelope, but may need further refinement.

GENERALIZATION OF THE *KGJ* MODEL AND $\overline{KGJ}$ MODEL

The formulations in Eqs. (1) and (2) are valid for triaxial stress state and need to generalized into a 3-D stress state before. Therefore, the present generalization of the formulations in Eqs. (1) and (2) is of theoretical and practical significance. This section discusses how to generalize Eqs. (1) and (2) for a 3-D stress state based on isotropic hypoelasticity. The general form of hypoelastic constitutive laws were proposed by Truesdell (1955, 1965). For time-independent materials, special forms of hypoelasticity were proposed by Rivlin and Ericksen (1955), Truesdell (1965) and Coon and Evans (1971). Rivlin and Ericksen (1955) derived hypoelastic relationships, by employing the Cayley-Hamilton Theorem, for incremental isotropic hypoelastic materials.

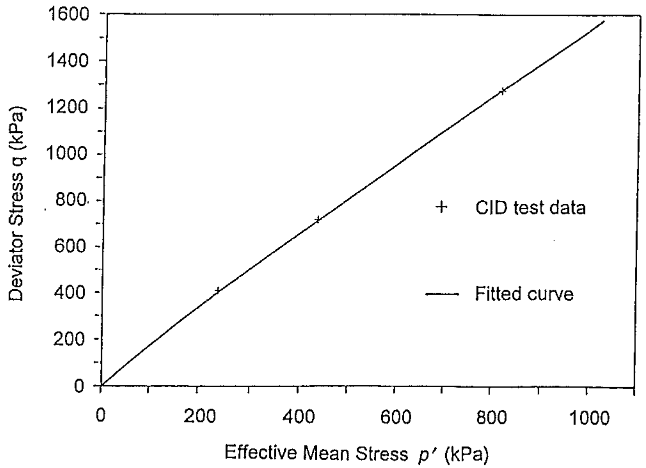


Fig. 4 Measured Data and Fitted Curve of q vs p' in CID and CIU Tests

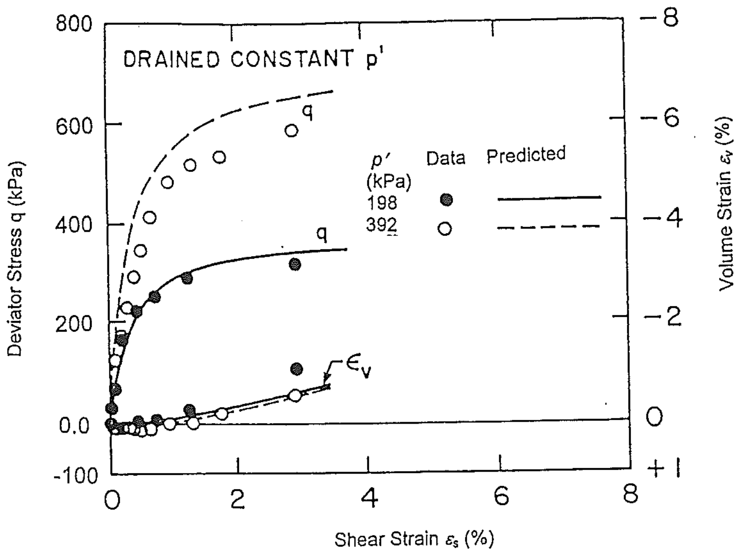


Fig. 5 Measured and Predicted Curves of q vs ε_s and ε_v vs ε_s in Constant - p' Shear Tests (Yin et al., 1989)

Here two special forms are discussed (Coon and Evans 1971, Chen and Mizuno, 1990). The first form is:

$$d\varepsilon_{ij} = C_{ijkl}(\sigma'_{mn})d\sigma'_{kl} \tag{21}$$

where C_{ijkl} is called tangential compliance tensor. Under the condition of incremental isotropy of material, C_{ijkl} becomes:

$$\begin{aligned} C_{ijkl}(\sigma'_{mn}) = {} & a_1\delta_{ij}\delta_{kl} + a_2(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}) + a_3\sigma'_{ij}\delta_{kl} + a_4\delta_{ij}\sigma'_{kl} + \\ & + a_5(\delta_{ik}\sigma'_{jl} + \delta_{il}\sigma'_{jk} + \delta_{jk}\sigma'_{il} + \delta_{jl}\sigma'_{ik}) + a_6\delta_{ij}\sigma'_{km}\sigma'_{ml} + a_7\delta_{kl}\sigma'_{im}\sigma'_{mj} \\ & + a_8(\delta_{ik}\sigma'_{jm}\sigma'_{ml} + \delta_{il}\sigma'_{jm}\sigma'_{mk} + \delta_{jk}\sigma'_{im}\sigma'_{ml} + \delta_{jl}\sigma'_{im}\sigma'_{mk}) + a_9\sigma'_{ij}\sigma'_{kl} + \\ & + a_{10}\sigma'_{ij}\sigma'_{km}\sigma'_{ml} + a_{11}\sigma'_{im}\sigma'_{mj}\sigma'_{kl} + a_{12}\sigma'_{im}\sigma'_{mj}\sigma'_{kn}\sigma'_{nl} \end{aligned} \tag{22}$$

In Eq. (22) a_1, a_2, ... a_{12} are 12 material coefficients which may depend on invariants of the effective stress tensor σ'_{ij}. The δ in Eq. (22) is the Kronecker delta (δ_{ij}= 1 for i = j or δ_{ij}= 0 for i $\neq$ j).

The second special form of isotropic hypoelasticity can be written as:

$$d\sigma'_{ij} = D_{ijkl}(\sigma'_{mn})d\varepsilon_{kl} \tag{23}$$

where D_{ijkl} is called tangential stiffness tensor. The form of D_{ijkl} (using b_1, b_2, ..., b_{12} instead of a_1 , a_2, ..., a_{12}) is the same as C_{ijkl} as in Eq. (22). Equations (21), (22) and (23) are the basis of the generalization of the two three-moduli models for triaxial stress state.

It is found that Eqs. (1) or (2) has only three independent coefficients (moduli). While Eq. (21) with C_{ijkl} in Eq. (22) or equation Eq. (23) has 12 independent coefficients. Simplification and assumptions must be made in order to generalize Eqs. (1) or (2) for a 3-D stress state. The principle of the generalization is:

(a) the generalized form can be reduced to (1) or (2) for a triaxial stress state and
(b) the coefficients (moduli) in the generalized form can be uniquely determined using known moduli K,G,J or $\overline{K},\overline{G},\overline{J}$.

Based on Eq. (1): Using this principle, the coefficients a_5, a_6, ..., a_{12} are assumed to be zero. Otherwise, they can not be uniquely determined. Thus, Eq. (22) is reduced as:

$$C_{ijkl}(\sigma'_{mn}) = a_1\delta_{ij}\delta_{kl} + a_2(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}) + a_3\sigma'_{ij}\delta_{kl} + a_4\delta_{ij}\sigma'_{kl} \tag{24}$$

Using Eq. (24), Eq. (21) can be written as, noting $\sigma'_{ij} = \sigma'_{ji}, d\varepsilon_{ij} = d\varepsilon_{ji}$:

$$d\varepsilon_{ij} = a_1\delta_{ij}d\sigma'_{kk} + 2a_2 d\sigma'_{ij} + a_3\sigma'_{ij}d\sigma_{kk} + a_4\delta_{ij}\sigma'_{kl}d\sigma'_{kl} \tag{25}$$

If two subscript indexes are the same, summation is implied, for example $d\varepsilon_{kk} = d\varepsilon_{11} + d\varepsilon_{22} + d\varepsilon_{33}$. The triaxial stress states are (a) all shear stresses and shear strains are zero and (b) the minor and middle principle stresses and strains are equal, that is,

$$\begin{Bmatrix} \sigma'_{ij} = 0, d\sigma'_{ij} = 0 \quad if\, i \neq j \\ \varepsilon_{ij} = 0, d\varepsilon_{ij} = 0 \quad if\, i \neq j \\ \sigma'_{22} = \sigma'_{33}, d\sigma'_{22} = d\sigma'_{33} \\ \varepsilon_{22} = \varepsilon_{33}, d\varepsilon_{22} = d\varepsilon_{33} \end{Bmatrix} \tag{26}$$

Under the triaxial condition in Eq. (26), from Eq. (25), we have:

$$\begin{Bmatrix} d\varepsilon_{11} = a_1 d\sigma'_{kk} + 2a_2 d\sigma'_{11} + a_3\sigma'_{11}d\sigma'_{kk} + a_4(\sigma'_{11}d\sigma'_{11} + 2\sigma'_{22}d\sigma'_{22}) \\ d\varepsilon_{22} = a_1 d\sigma'_{kk} + 2a_2 d\sigma'_{22} + a_3\sigma'_{22}d\sigma'_{kk} + a_4(\sigma'_{11}d\sigma'_{11} + 2\sigma'_{22}d\sigma'_{22}) \end{Bmatrix} \tag{27}$$

Noting that:

$$\begin{Bmatrix} \sigma'_{11} = p' + \tfrac{2}{3}q, & \sigma'_{22} = p' - \tfrac{1}{3}q \\ d\sigma'_{11} = dp' + \tfrac{2}{3}dq, & d\sigma'_{22} = dp' - \tfrac{1}{3}dq \\ \varepsilon_{11} = \tfrac{1}{3}\varepsilon_v + \varepsilon_s, & \varepsilon_{22} = \tfrac{1}{3}\varepsilon_v - \tfrac{1}{2}\varepsilon_s \\ d\varepsilon_{11} = \tfrac{1}{3}d\varepsilon_v + d\varepsilon_s, & d\varepsilon_{22} = \tfrac{1}{3}d\varepsilon_v - \tfrac{1}{2}d\varepsilon_s \end{Bmatrix} \tag{28}$$

Eq. (27) can be written as:

$$\begin{Bmatrix} d\varepsilon_v = (9a_1 + 6a_2 + 9p'a_3 + 9p'a_4)dp' + 2a_4 q dq \\ d\varepsilon_s = 2a_3 q dp' + \tfrac{4}{3}a_2 dq \end{Bmatrix} \tag{29}$$

Comparing Eq. (29) with Eq. (1), we can determine:

$$a_4 = a_3 = \frac{1}{2qJ}, \quad a_2 = \frac{1}{4G}, \quad a_1 = \frac{1}{9K} - \frac{1}{6G} - \frac{p'}{qJ} \tag{30}$$

Based on Eq. (2): Again using the same principle as mentioned before, the coefficients b_5, b_6, ..., b_{12} are assumed to be zero. Thus, Eq. (23) with D_{ijkl} similar to C_{ijkl} in Eq. (22) is reduced to:

$$D_{ijkl}(\sigma'_{mn}) = b_1\delta_{ij}\delta_{kl} + b_2(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}) + b_3\sigma'_{ij}\delta_{kl} + b_4\delta_{ij}\sigma'_{kl} \tag{31}$$

Using Eq. (31), Eq. (23) can be written as, noting $\sigma'_{ij} = \sigma'_{ji}, d\varepsilon_{ij} = d\varepsilon_{ji}$:

$$d\sigma'_{ij} = b_1\delta_{ij}d\varepsilon_{kk} + 2b_2 d\varepsilon_{ij} + b_3\sigma'_{ij}d\varepsilon_{kk} + b_4\delta_{ij}\sigma'_{kl}d\varepsilon_{kl} \tag{32}$$

Under the triaxial condition in Eq. (26), from Eq. (32), we have:

$$\begin{Bmatrix} d\sigma'_{11} = b_1 d\varepsilon_{kk} + 2b_2 d\varepsilon_{11} + b_3\sigma'_{11}d\varepsilon_{kk} + b_4(\sigma'_{11}d\varepsilon_{11} + 2\sigma'_{22}d\varepsilon_{22}) \\ d\sigma'_{22} = b_1 d\varepsilon_{kk} + 2b_2 d\varepsilon_{22} + b_3\sigma'_{22}d\varepsilon_{kk} + b_4(\sigma'_{11}d\varepsilon_{11} + 2\sigma'_{22}d\varepsilon_{22}) \end{Bmatrix} \tag{33}$$

Noting the relationships in Eq. (28), equation Eq. (33) can be written as:

$$\begin{Bmatrix} dp' = (b_1 + \tfrac{2}{3}b_2 + p'b_3 + p'b_4)\varepsilon_v + b_4 q\varepsilon_s \\ dq = b_3 q\varepsilon_v + 3b_2\varepsilon_s \end{Bmatrix} \tag{34}$$

Comparing Eq. (34) with Eq. (2), we can determine:

$$b_4 = b_3 = -\frac{\bar{J}}{q}, \quad b_2 = \bar{G}, \quad b_1 = \bar{K} - \frac{2}{3}\bar{G} + \frac{2p'}{q}\bar{J} \tag{35}$$

Using Eqs. (30) and (35), the two tensor forms of Eqs. (25) and (33) can be written as:

$$d\varepsilon_{ij} = \left(\frac{1}{9K} - \frac{1}{6G} - \frac{p'}{qJ}\right)\delta_{ij}d\sigma'_{kk} + \frac{1}{2G}d\sigma'_{ij} + \frac{1}{2qJ}\sigma'_{ij}d\sigma'_{kk} + \frac{1}{2qJ}\delta_{ij}\sigma'_{kl}d\sigma'_{kl} \tag{36}$$

$$d\sigma'_{ij} = \left(\bar{K} - \frac{2}{3}\bar{G} + \frac{2p'}{q}\bar{J}\right)\delta_{ij}d\varepsilon_{kk} + 2\bar{G}d\varepsilon_{ij} - \frac{\bar{J}}{q}\sigma'_{ij}d\varepsilon_{kk} - \frac{\bar{J}}{q}\delta_{ij}\sigma'_{kl}d\varepsilon_{kl} \tag{37}$$

Noting that in a general stress state, the mean stress p' and the deviator stress q are:

$$\begin{Bmatrix} p' = \sigma'_{kk}/3 = (\sigma'_{11} + \sigma'_{22} + \sigma'_{33})/3 \\ q = \sqrt{\frac{3}{2}}(S_{ij}S_{ij})^{1/2} = \frac{1}{\sqrt{2}}\left[(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 + 6(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2)\right]^{1/2} \end{Bmatrix} \tag{38}$$

The matrix form of Eq. (36) can be written as $\{d\varepsilon\} = [C]\{d\sigma'\}$, that is:

$$\begin{Bmatrix} d\varepsilon_{11} \\ d\varepsilon_{22} \\ d\varepsilon_{33} \\ d\gamma_{12} \\ d\gamma_{23} \\ d\gamma_{31} \end{Bmatrix} = \begin{bmatrix} \alpha_1 + 2\alpha_3 & \alpha_2 + \alpha_3 + \alpha_4 & \alpha_2 + \alpha_3 + \alpha_5 & \frac{\sigma_{12}}{qJ} & \frac{\sigma_{23}}{qJ} & \frac{\sigma_{31}}{qJ} \\ \alpha_2 + \alpha_3 + \alpha_4 & \alpha_1 + 2\alpha_4 & \alpha_2 + \alpha_4 + \alpha_5 & \frac{\sigma_{12}}{qJ} & \frac{\sigma_{23}}{qJ} & \frac{\sigma_{31}}{qJ} \\ \alpha_2 + \alpha_3 + \alpha_5 & \alpha_2 + \alpha_4 + \alpha_5 & \alpha_1 + 2\alpha_5 & \frac{\sigma_{12}}{qJ} & \frac{\sigma_{23}}{qJ} & \frac{\sigma_{31}}{qJ} \\ \frac{\sigma_{12}}{qJ} & \frac{\sigma_{12}}{qJ} & \frac{\sigma_{12}}{qJ} & \frac{1}{G} & 0 & 0 \\ \frac{\sigma_{23}}{qJ} & \frac{\sigma_{23}}{qJ} & \frac{\sigma_{23}}{qJ} & 0 & \frac{1}{G} & 0 \\ \frac{\sigma_{31}}{qJ} & \frac{\sigma_{31}}{qJ} & \frac{\sigma_{31}}{qJ} & 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{Bmatrix} d\sigma'_{11} \\ d\sigma'_{22} \\ d\sigma'_{33} \\ d\sigma_{12} \\ d\sigma_{23} \\ d\sigma_{31} \end{Bmatrix} \tag{39}$$

where $d\gamma_{12} = 2\varepsilon_{12}, d\gamma_{23} = 2\varepsilon_{23}, d\gamma_{31} = 2\varepsilon_{31}, \alpha_1 = \frac{1}{9K} + \frac{1}{3G}, \alpha_2 = \frac{1}{9K} - \frac{1}{6G}, \alpha_3 = \frac{2\sigma'_{11} - \sigma'_{22} - \sigma'_{33}}{6qJ}$, $\alpha_4 = \frac{2\sigma'_{22} - \sigma'_{11} - \sigma'_{33}}{6qJ}, \alpha_5 = \frac{2\sigma'_{33} - \sigma'_{11} - \sigma'_{22}}{6qJ}$. It is seen that the matrix [C] in Eq. (39) is symmetric. If no shear dilation, *i.e.* $J=\infty$, all items multiplied by $1/J$ will be zero and there are only α_1, α_2 and $1/G$ left in the matrix which is the same as that for an isotropic elasticity matrix.

The matrix form of Eq. (37) can be written as $\{d\sigma'\} = [D]\{d\varepsilon\}$, that is:

$$\begin{Bmatrix} d\sigma'_{11} \\ d\sigma'_{22} \\ d\sigma'_{33} \\ d\sigma_{12} \\ d\sigma_{23} \\ d\sigma_{31} \end{Bmatrix} = \begin{bmatrix} \beta_1 + 2\beta_3 & \beta_2 + \beta_3 + \beta_4 & \beta_2 + \beta_3 + \beta_5 & \frac{-\bar{J}\sigma_{12}}{q} & \frac{-\bar{J}\sigma_{23}}{q} & \frac{-\bar{J}\sigma_{31}}{q} \\ \beta_2 + \beta_3 + \beta_4 & \beta_1 + 2\beta_4 & \beta_2 + \beta_4 + \beta_5 & \frac{-\bar{J}\sigma_{12}}{q} & \frac{-\bar{J}\sigma_{23}}{q} & \frac{-\bar{J}\sigma_{31}}{q} \\ \beta_2 + \beta_3 + \beta_5 & \beta_2 + \beta_4 + \beta_5 & \beta_1 + 2\beta_5 & \frac{-\bar{J}\sigma_{12}}{q} & \frac{-\bar{J}\sigma_{23}}{q} & \frac{-\bar{J}\sigma_{31}}{q} \\ \frac{-\bar{J}\sigma_{12}}{q} & \frac{-\bar{J}\sigma_{12}}{q} & \frac{-\bar{J}\sigma_{12}}{q} & \bar{G} & 0 & 0 \\ \frac{-\bar{J}\sigma_{23}}{q} & \frac{-\bar{J}\sigma_{23}}{q} & \frac{-\bar{J}\sigma_{23}}{q} & 0 & \bar{G} & 0 \\ \frac{-\bar{J}\sigma_{31}}{q} & \frac{-\bar{J}\sigma_{31}}{q} & \frac{-\bar{J}\sigma_{31}}{q} & 0 & 0 & \bar{G} \end{bmatrix} \begin{Bmatrix} d\varepsilon_{11} \\ d\varepsilon_{22} \\ d\varepsilon_{33} \\ d\gamma_{12} \\ d\gamma_{23} \\ d\gamma_{31} \end{Bmatrix} \tag{40}$$

where $\beta_1 = \bar{K} + \frac{4}{3}\bar{G}, \beta_2 = \bar{K} - \frac{2}{3}\bar{G}, \beta_3 = \frac{\bar{J}}{3q}(\sigma'_{22} + \sigma'_{33} - 2\sigma'_{11}), \beta_4 = \frac{\bar{J}}{3q}(\sigma'_{11} + \sigma'_{33} - 2\sigma'_{22}),$. It is seen from Eq. (40) that the matrix [D] is symmetric. If there is no shear-dilation, that is $\bar{J} = 0$, all the items multiplied by $\bar{J}$ become zero. The resulting matrix is the same as that for isotropic elasticity matrix.

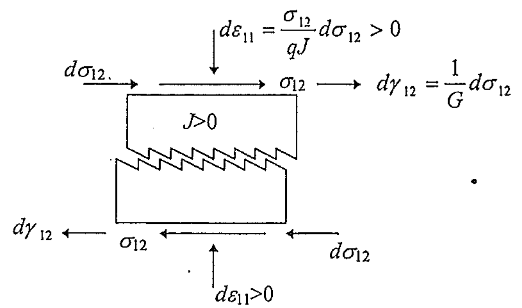
It is noted that in Eqs. (39) and (40), the items with $1/J$ or $\bar{J}$ (including $\alpha_3, \alpha_4, \alpha_5$ and $\beta_3, \beta_4, \beta_5$) are divided by the deviator stress q. If $q = 0$, all those items divided by q shall be set to zero. When $q > 0$ and $1/J$ or $\bar{J} \neq 0$, the behavior simulated is "induced anisotroic". The matrix [C] in Eq. (39) can be inversed as $[C]^{-1}$ to express $\{d\sigma\}$ in terms of $\{d\varepsilon\}$. However, $[C]^{-1}$ may not be equal to [D] in Eq. (40). The same is true that $[D]^{-1}$ may not be equal to [C] in Eq. (39).

PHYSICAL MEANINGS OF ITEMS IN THE TWO GENERALIZED FORMS

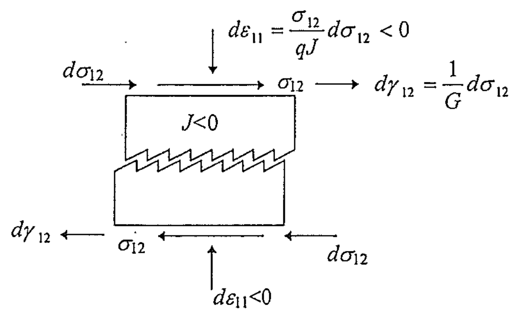
From Eq. (39), under pure shear stress increment $d\sigma_{12}$ only, the shear stress induced strains are:

$$d\varepsilon_{11}=\frac{\sigma_{12}}{qJ}d\sigma_{12},\ d\varepsilon_{22}=\frac{\sigma_{12}}{qJ}d\sigma_{12}$$
$$d\varepsilon_{33}=\frac{\sigma_{12}}{qJ}d\sigma_{12},\ d\gamma_{12}=\frac{1}{G}d\sigma_{12} \tag{41}$$

The shear strain increment $d\gamma_{12}$ caused by $d\sigma_{12}$ is expected. If the pre-shear stress σ_{12} exists, that is, $\sigma_{12}>0$ and $q>0$, then the shear stress increment $d\sigma_{12}$ will produce axial strain increments $d\varepsilon_{11}, d\varepsilon_{22}$, $d\varepsilon_{33}$ in three directions controlled by the coupling modulus J. The shear stress induced axial strains are illustrated in Fig. 6(a) for shear-compression behavior (loose sand) ($J>0$) and in Fig. 6(b) for shear-dilation (expansion) behavior (dense sand) ($J<0$). For shear-compression behavior, the resulting axial strain increment $d\varepsilon_{11}$ is positive (downward) in Fig.6(a). But the resulting axial strain increment $d\varepsilon_{11}$ is negative (upward) as shown in Fig.6(b) for shear-dilation behavior. This is consistent with observations.



(a)



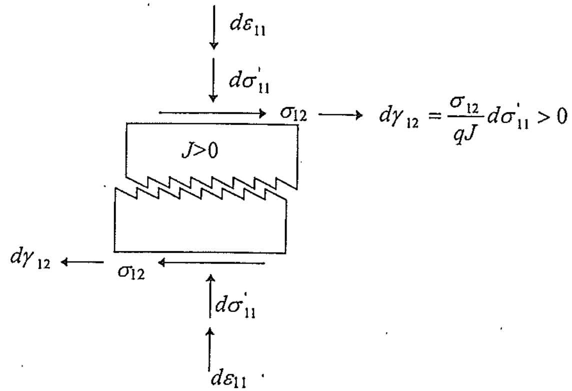
(b)

Fig. 6 Under Shear Stress $d\sigma_{12}$ only, (a) Shear-Compression Behavior, (b) Shear-Dilation Behavior

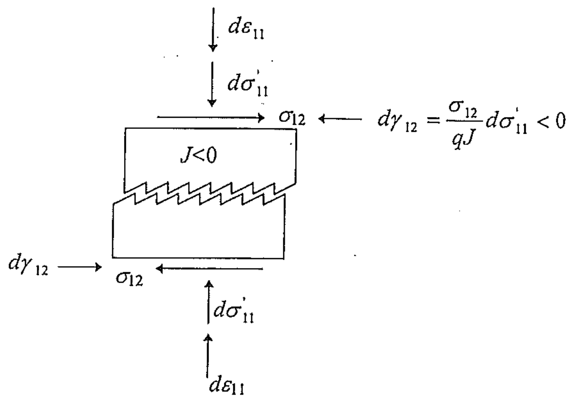
From Eq. (39), under pure axial stress increment $d\sigma_{11}$ only, the axial stress induced strains are:

$$d\varepsilon_{11}=\left(\frac{1}{9K}+\frac{1}{3G}+\frac{2\sigma'_{11}-\sigma'_{22}-\sigma'_{33}}{3qJ}\right)d\sigma'_{11}$$
$$d\gamma_{12}=\frac{\sigma_{12}}{qJ}d\sigma'_{11},\ d\gamma_{23}=\frac{\sigma_{23}}{qJ}d\sigma'_{11},\qquad d\gamma_{31}=\frac{\sigma_{31}}{qJ}d\sigma'_{11} \tag{42}$$

The axial strain increment $d\varepsilon_{11}$ resulted due to $d\sigma'_{11}$ is expected, but depending on $(2\sigma'_{11}-\sigma'_{22}-\sigma'_{33})/(3qJ)$. If pre-shear stresses σ_{12}, σ_{23}, and σ_{31} exist, then the axial stress increment $d\sigma'_{11}$ will produce shear strain increments $d\gamma_{11}, d\gamma_{23}$, $d\gamma_{31}$ with respect to σ_{12}, σ_{23}, and σ_{31} accordingly, controlled by the coupling modulus J. The axial stress induced shear strain are illustrated in Fig. 7(a) for shear-compression behavior (loose sand) ($J>0$) and in Fig. 7(b) for shear-dilation (expansion) behavior (dense sand) ($J<0$). For shear-compression behavior, the resulting shear strain increment $d\gamma_{12}$ is positive (toward right) in Fig. 7(a). But the resulting shear strain increment $d\gamma_{12}$ is negative (toward left) as shown in Fig. 7(b) for shear-dilation behavior.



(a)



(b)

Fig. 7 Under Axial Stress $d\sigma'_{11}$ only, (a) Shear-Compression Behavior, (b) Shear-Dilation Behavior

From Eq. (40), under pure shear strain increment $d\gamma_{12}$ only, the shear strain induced stresses are:

$$d\sigma'_{11} = \frac{-\bar{J}\sigma_{12}}{q} d\gamma_{12}, \quad d\sigma'_{22} = \frac{-\bar{J}\sigma_{12}}{q} d\gamma_{12}$$
$$d\sigma'_{33} = \frac{-\bar{J}\sigma_{12}}{q} d\gamma_{12}, \quad d\sigma_{12} = \bar{G} d\gamma_{12} \tag{43}$$

It is expected that the shear strain $d\gamma_{12}$ produces shear stress increment $d\sigma_{12}$. If the pre-shear stress σ_{12} exists, then the shear strain increment $d\gamma_{12}$ will produce axial stress increments $d\sigma'_{11}$, $d\sigma'_{22}$, $d\sigma'_{33}$ in three directions, negative for shear-compression behavior with $\bar{J} > 0$ as shown in Fig. 8(a), but positive for shear-dilation behavior with $\bar{J} < 0$ as in Fig. 8(b). For a shear-compression soil (loose sand), the shear strain increment $d\gamma_{12}$ will cause compression (collapse) of the soil as illustrated in Fig. 8(a), therefore a reduction in the axial stress. However, for a shear-dilation soil (dense sand), the shear strain increment $d\gamma_{12}$ will cause expansion (additional vertical constrain) of the soil as illustrated in Fig. 8(b), therefore an increase in the axial stress. These modeling results are considered consistent with the physical behavior of shear-compression or shear-dilation soils.

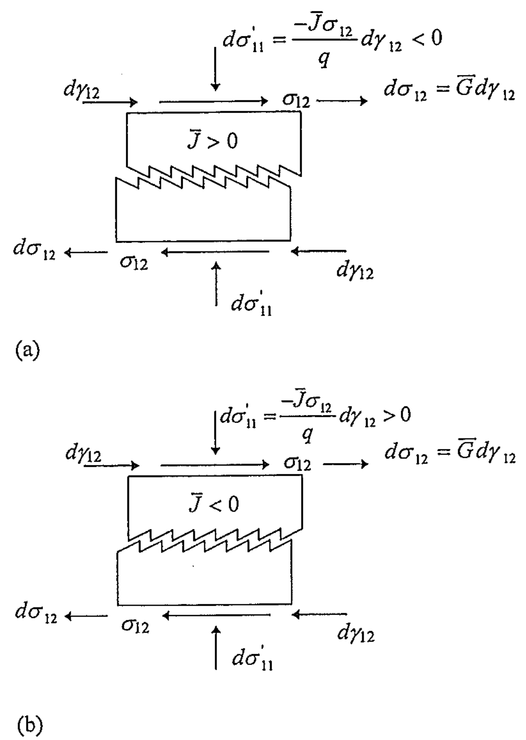


Fig. 8 Under Shear Strain $d\gamma_{12}$ only, (a) Shear-Compression Behavior, (b) Shear Dilation Behavior

From Eq. (40), under pure axial strain increment $d\varepsilon_{11}$ only, the axial strain increment induced stress increments are

$$d\sigma'_{11} = [\bar{K} + \frac{4}{3}\bar{G} + \frac{2\bar{J}}{3q}(\sigma'_{22} + \sigma'_{33} - 2\sigma'_{11})] d\varepsilon_{11}$$
$$d\sigma_{12} = \frac{-\bar{J}\sigma_{12}}{q} d\varepsilon_{11}, \quad d\sigma_{23} = \frac{-\bar{J}\sigma_{23}}{q} d\varepsilon_{11}, \quad d\sigma_{31} = \frac{-\bar{J}\sigma_{31}}{q} d\varepsilon_{11} \tag{44}$$

It is expected that an axial stress increment $d\sigma'_{11}$ is produced by $d\varepsilon_{11}$. However, the resulting $d\sigma'_{11}$ depends on $2\bar{J}(\sigma'_{22} + \sigma'_{33} - 2\sigma'_{11})/(3q)$. If pre-shear stresses σ_{12}, σ_{23}, and σ_{31} exist, then the axial strain increment $d\varepsilon_{11}$ will produce shear stress increments $d\sigma_{12}$, $d\sigma_{23}$, $d\sigma_{31}$ with respect to σ_{12}, σ_{23}, and σ_{31} accordingly, controlled by the coupling modulus $\bar{J}$. The axial strain induced shear stresses are illustrated in Fig. 9(a) for shear-compression behavior ($\bar{J} > 0$) and in Fig. 9(b) for shear-dilation behavior ($\bar{J} < 0$). For a shear-compression soil (loose sand) as illustrated in Fig. 9(a), additional shear strain increment $d\gamma_{12}$ (towards right) can occur under the axial strain increment $d\varepsilon_{11}$. In other words, the upper half zig-zag "soil" has the potential to move towards right. Imagine a plate in contact with the top surface of the upper half zig-zag "soil", the plate will drag the "soil" towards left for the potential movement of the "soil" towards right, thereby reducing the shear stress σ_{12}, that is, resulting a negative $d\sigma_{12} < 0$. The same explanation can be used for a shear-dilation soil as shown in Fig. 9(b).

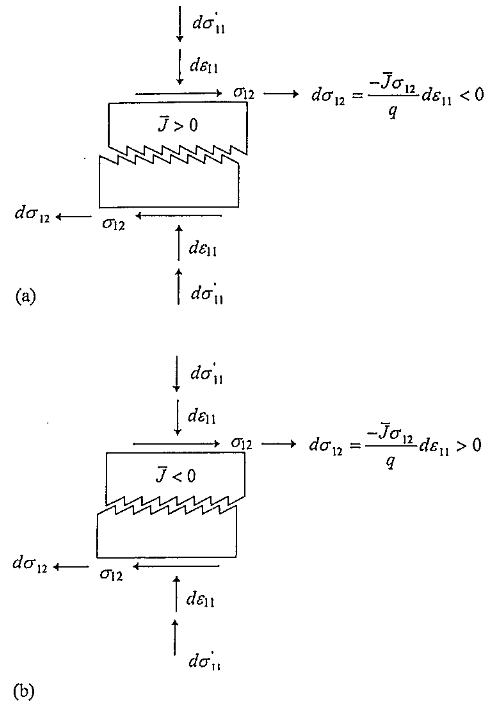


Fig. 9 Under Axial Strain $d\varepsilon_{11}$ only, (a) Shear-Compression Behavior, (b) Shear-Dilation Behavior

DISCUSSIONS

The proposed *KGJ* model is an improvement of the two-moduli models (Duncan and Chang, 1970; Domaschuk and Villiappan, 1975) which cannot consider shear stress induced dilation or compression. In spite of the improvement, the *KGJ* model has its limitations. For example, the *KGJ* model has the same problem as the two-moduli models in defining an unloading/reloading condition. This issue has not been investigated here. The *KGJ* model may use the same unloading/reloading criteria as used by Duncan and Chang (1970). The curve fitting functions used to determine the three moduli in the *KGJ* model in this paper may not be appropriate for some soils. However, these functions may be replaced by any other mathematical functions for best data fitting based on the new idea and framework of the *KGJ* model. For example, if the stress-strain behavior shows strain-softening with a peak shear strength, a proper mathematical function shall be selected to fit the stress-strain curve with a peak value and likely a residual value at a large strain. The generalized form of the *KGJ* model (or $\overline{KGJ}$ model) has a clear physical meaning and has shown the potential to describe the non-linear and shear-dilative stress-strain behavior of soils.

CONCLUSIONS

The prediction using the *KGJ* model calibrated using CID test data is in good agreement with the measured test data from three drained triaxial shear tests in constant p'. It is found that the two generalized matrixes are symmetric and therefore are easy for implementation in finite element analyses. It is also found that the generalized relationships can explain the physical phenomena for shear-compression soils or shear-dilation soils.

The verification of the generalized models is important. However, the true and complete verification of the generalized model by test data is not easy or probably impossible since data of tests with independent control of six stresses may be necessary. Limited direct verification may be done for example using data from a hollow cylinder apparatus. Indirect verification may be done by implementing the generalized models in finite element analysis of geotechnical structures, for example, a footing on soils and comparing predicted results with measured results. Further research in this area is necessary.

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