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VIBRATION OF MACHINE FOUNDATIONS ON ELASTIC MEDIA

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SYNOPSIS

Vibrations of a long machine foundation are studied for which the excitation force caused by the unbalanced mass of the machine is frequency dependent. The medium supporting the foundation is assumed to be isotropic and elastic. The three modes of vibration involved are vertical, horizontal and rocking; the horizontal and rocking motions are coupled for this analysis. The response curves for the vertical motion contain one resonant peak, while two resonant peaks exist in the coupled horizontal and rocking motion. It is found that the foundation behavior depends heavily upon the operating frequency, foundation mass and dimensions, the Poisson's ratio of the medium having less influence. The results are presented in the form of charts to facilitate the analysis and dedign of actual foundations.

INTRODUCTION

The vibration of a thick spread footing supporting a machine can be treated, for practical purposes, as that of a rigid body on a half-space. Many investigations have been carried out to obtain solutions for the vibration of a rigid body on an elastic, homogeneous and isotropic half-space. The approaches adopted in these studies can be classified into two categories; either a suitable contact stress distribution is assumed or the indentation of the rigid body is specified.

For three dimensional problems solutions have been given when oscillating stresses are prescribed in a circular region. REISSNER (1936) and later MILLER & PURSEY (1954) treated the case of a uniform pressure on an elastic half-space. SUNG (1953), QUINLAN (1953) and BYCROFT (1956) approached two- and three-dimensional problems by assuming the dynamic stress distribution to be proportional to the static stress distribution. RICHART et al (1960, 1966, 1967 and 1967) treated three-dimensional problems by using a mass-spring system and a mass-spring-dashpot system and compared their solutions with the theoretical solutions of SUNG (1953) and BYCROFT (1956) and with the test results of FRY (1963).

When a body vibrates on an elastic half-space, the problem may be for-

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mulated so as to be governed by a set of dual integral equations, the first one corresponding to the specified displacement given by the body while the second represents the zero stress condition outside of the contact region. AWOJOBI & GROOTENHUIS (1965) and ROBERTSON (1966) solved the problem of a smooth, circular, rigid disk undergoing vertical oscillations, while ZAKORKO & ROSTOVTSEV (1965) treated the cases of vertical and rocking oscillations. Analogous to these three-dimensional problems are the two-dimensional cases, which are also governed by a set of dual integral equations. AWOJOBI (1966) solved the problem of rocking vibration. KARASUDHI et al (1968) treated the cases of vertical, horizontal and rocking oscillations.

All the investigations mentioned in the foregoing involved a body undergoing vibrations with a single degree of freedom. Using a suitable superposition, KARASUDHI et al (1968) gave an approximate relationship between contact forces and displacements of coupled horizontal and rocking vibrations.

The purpose of this research is to study the behavior of a long, rectangular, rigid machine foundation resting on an elastic medium. The vibration is caused by an unbalanced mass in the machine rotating in a vertical plane and about a horizontal axis located at a certain height above the surface of the medium. The stiffnesses for vertical and coupled horizontal and rocking vibrations proposed by KARASUDHI et al (1968) are used.

METHOD OF ANALYSIS

The vibration of a rigid, rectangular foundation of infinite length resting on an isotropic elastic half-space is considered. Figure 1 shows the coordinate system and the significant dimensions[1]. The infinite length of the foundation runs along the z-axis and the half-space occupies the region $y \geq O$. The system is excited by an unbalanced mass m_o per unit length along the z-axis, which rotates in the xy-plane with a constant angular velocity Ω and an eccentricity ε. Since m_o is small compared to the total mass of the system, the motion of the axis of rotation of m_o is proportionally small in comparison with ε. The force applied to the system, for practical purposes, is approximately equal to a centrifugal force of magnitude

$$F = m_o \Omega^2 \varepsilon \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

A rigid element of height H rigidly connects the axis of rotation of m_o and the mass center of the rigid foundation. Harmonic motions are assumed in this study and the mass center of the foundation is assumed to coincide with

[1] A complete list of the symbols used is given in the Appendix.

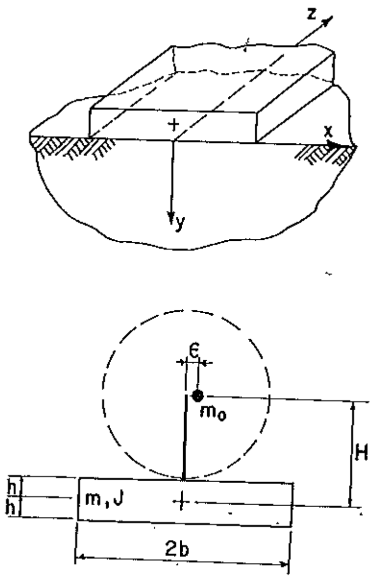

Fig. 1. The half-space model.

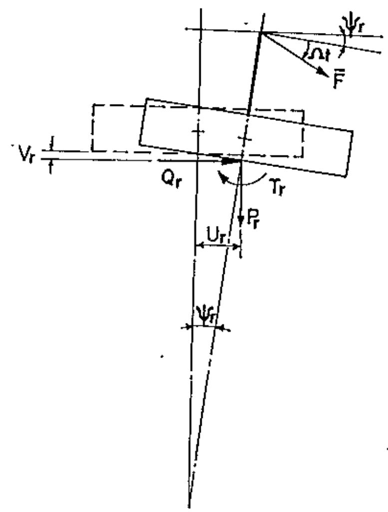

Fig. 2. Displaced configuration.

the geometric centroid of the cross-section of the machine-foundation system.

Under the applied force given by Eq. (1), the center of the base of the foundation undergoes vertical displacement v_r, horizontal translation u_r and rocking angle Ψ_r. These three modes of vibration are depicted in Fig. 2 and are governed by the three equations of motion

$$m \frac{d^2}{dt^2}(v_r + h - h \cos \Psi_r) = P_r + F \sin(\Omega t + \Psi_r), \quad \ldots\ldots \quad (2)$$

$$m \frac{d^2}{dt^2}(u_r + h \sin \Psi_r) = Q_r + F \cos(\Omega t + \Psi_r), \quad \ldots\ldots \quad (3)$$

$$J \frac{d^2}{dt^2}(\Psi_r) = T_r - hQ_r \cos \Psi_r - hP_r \sin \Psi_r + HF \cos(\Omega t + \Psi_r) \quad \ldots \quad (4)$$

where m denotes the mass of the foundation per unit length along the z-axis, J is the mass polar moment of inertia about the mass center per unit length of the foundation along the z-axis, $2b$ and $2h$ are respectively the width and thickness of the foundation, and P_r, Q_r and T_r are respectively the contact vertical force, contact horizontal force and contact torque per unit length along the z-axis.

In harmonic motions, the contact forces and torque are given in terms of the displacements by KARASUDHI et al (1968) in the form

$$\mathbf{P} = \pi G \bar{a} \mathrm{v}, \quad \ldots\ldots\ldots\ldots (5)$$

$$\begin{Bmatrix} \mathbf{Q} \\ \mathbf{T}/b \end{Bmatrix} = \pi G \begin{bmatrix} \bar{a}_{11} & \bar{a}_{12} \\ \bar{a}_{21} & \bar{a}_{22} \end{bmatrix} \begin{Bmatrix} \mathrm{u} \\ b\psi \end{Bmatrix} \ldots\ldots\ldots\ldots (6)$$

where Re(**P**), Re (**Q**) and Re(**T**) denote the real parts of the complex functions in the parentheses and are equal to P_r, Q_r and T_r respectively, Re (v), Re (u) and Re (ψ) are equal to v_r, u_r and Ψ_r respectively, G denotes the modulus of rigidity of the elastic medium, and $\bar{a}$, $\bar{a}_{11}$, $\bar{a}_{12}$, $\bar{a}_{21}$ and $\bar{a}_{22}$ are complex stiffnesses. It should be noted that $\bar{a}_{12} = \bar{a}_{21}$.

The complex displacement functions v, u and ψ, which are assumed to be harmonic, can be represented in the forms

$$\mathrm{v} = \mathrm{V} \exp\,[i\,(\Omega t + \delta_1)], \ldots\ldots\ldots\ldots (7)$$

$$\mathrm{u} = \mathrm{U} \exp\,[i\,(\Omega t + \delta_2)], \ldots\ldots\ldots\ldots (8)$$

$$\psi = \overline{\Psi} \exp\,[i\,(\Omega t + \delta_3)], \ldots\ldots\ldots\ldots (9)$$

where V, U and $\overline{\Psi}$ are real functions which depend on frequency, and δ_1, δ_2 and δ_3 denote the phase angles. On the assumption that V, U and $\overline{\Psi}$ are small Eqs. (7), (8) and (9) lead to

$$\cos(\Psi_r) \approx 1, \quad \sin(\Psi_r) \approx \Psi_r, \ldots\ldots\ldots\ldots (10)$$

$$\sin(\Omega t + \Psi_r) \approx \sin(\Omega t), \quad \cos(\Omega t + \Psi_r) \approx \cos(\Omega t), \ldots (11)$$

$$\frac{d^2}{dt^2} \cos(\Psi_r) \approx 0, \quad \frac{d^2}{dt^2} \sin(\Psi_r) \approx -\Omega^2 \,\mathrm{Re}\,(\psi), \quad \mathrm{Re}\,(\bar{a}\mathrm{v})\,\Psi_r \approx 0 \ldots (12)$$

The use of Eq. (1) and Eqs. (5) to (12) results in Eqs. (2), (3) and (4) becoming

$$-\Omega^2 m \,\mathrm{Re}\,(\mathrm{v}) = \pi G \,\mathrm{Re}\,(\bar{a}\mathrm{v}) - \mathrm{Re}\,[i m_0\, \varepsilon \Omega^2 \exp\,(i\Omega t)] \ldots (13)$$

$$-\Omega^2 m \,\mathrm{Re}\,(\mathrm{u} + h\psi) = \pi G \,\mathrm{Re}\,(\bar{a}_{11}\mathrm{u} - \bar{a}_{12} b\psi) + \mathrm{Re}\,[m_0\, \varepsilon \Omega^2 \exp\,(i\Omega t)] \ldots\ldots\ldots\ldots (14)$$

$$-\Omega^2 J \,\mathrm{Re}\,(\psi) = \pi G b \,\mathrm{Re}\,(\bar{a}_{12}\mathrm{u} + \bar{a}_{22} b\psi) - \pi G h \,\mathrm{Re}\,(\bar{a}_{11}\mathrm{u} + \bar{a}_{12} b\psi) + H \,\mathrm{Re}\,[m_0\, \varepsilon \Omega^2 \exp\,(i\Omega t)] \ldots (15)$$

Substitution of Eqs. (7), (8) and (9) in Eqs. (13), (14) and (15) leads to three equations of the form

$$\mathrm{Re}\,[A \exp\,(i\Omega t)] = \mathrm{Re}\,[B \exp\,(i\Omega t)] \ldots\ldots\ldots (16)$$

where A and B are complex functions and can be shown by a simple expansion to be equal; Eqs. (13), (14) and (15) become, therefore,

$$(\overline{\mathrm{m}}\eta^2 + \pi\bar{a})\, \nu \exp\,(i\delta_1) = i\eta^2, \ldots\ldots\ldots\ldots (17)$$

$$(\overline{\mathrm{m}}\eta^2 + \pi\bar{a}_{11})\, u \exp\,(i\delta_2) + (\overline{\mathrm{m}}\eta^2\zeta_1 + \pi\bar{a}_{12})\, \Psi \exp\,(i\delta_3) = -\eta^2, \ldots (18)$$

$$\pi\,(-\zeta_1 \bar{a}_{11} + \bar{a}_{12})\, u \exp\,(i\delta_2) + [\overline{\mathrm{m}}\eta^2\,(1 + \zeta_1{}^2)/3 - \pi\,(\zeta_1 \bar{a}_{12} - \bar{a}_{22})]\, \Psi \exp\,(i\delta_3) = -\zeta_2\eta^2, \ldots\ldots (19)$$

where

$$(\nu, u, \Psi) = (\mathrm{V}, \mathrm{U}, b\overline{\Psi})\, \rho b^2/\varepsilon m_0, \ldots\ldots\ldots (20)$$

$$\overline{\mathrm{m}} = \frac{m}{\rho b^2} = \frac{3J}{\rho b^2\,(h^2 + b^2)}, \ldots\ldots\ldots\ldots (21)$$

$$\eta = \frac{b\Omega}{c}, \quad c = \sqrt{\frac{G}{\rho}}, \ldots\ldots\ldots\ldots (22)$$

$$\zeta_1 = h/b, \quad \zeta_2 = H/b \ldots\ldots\ldots\ldots (23)$$

In these equations ρ is the mass density of the elastic medium, and $\overline{\mathrm{m}}$ and η will, henceforth, be referred to as the 'mass ratio' and the 'frequency factor' respectively.

It is obvious that the vertical vibration of the foundation is governed by Eq. (17), and that the coupled horizontal and rocking vibrations are governed by Eqs. (18) and (19). In Eq. (22) c is the shear wave velocity of the elastic medium.

Vertical Vibration

Solution of the complex Eq. (17) yields

$$\nu = \eta^2 / \left\{ [\overline{\mathrm{m}}\eta^2 + \pi\,(\mathrm{Re}(\bar{a})]^2 + [\pi \,\mathrm{Im}\,(\bar{a})]^2 \right\}^{1/2}, \ldots\ldots (24)$$

$$\delta_1 = \mathrm{arc\ tan}\left\{ [\overline{\mathrm{m}}\eta^2 + \pi \,\mathrm{Re}\,(\bar{a})]/\pi \,\mathrm{Im}\,(\bar{a}) \right\}, \ldots\ldots (25)$$

where Im denotes the imaginary part of the complex function in the parentheses. The non-dimensional amplitude ν is plotted against arguments of η in Fig. 3 (at the end of the paper) for values of Poisson's ratio, υ, of 0, 1/4, 1/3 and 1/2, and for various values of $\overline{\mathrm{m}}$.

Coupled Horizontal and Rocking Vibrations

Solution of the complex Eqs. (18) and (19) yields

$$u = \frac{\sqrt{(A_1 D_1 + A_2 D_2)^2 + (A_1 D_2 - A_2 D_1)^2}}{D_1^2 + D_2^2}, \ldots\ldots (26)$$

$$\Psi = \frac{\sqrt{(B_1D_1 + B_2D_2)^2 + (B_1D_2 - B_2D_1)^2}}{D_1^2 + D_2^2}, \quad \ldots\ldots (27)$$

$$\delta_2 = \text{arc tan}\left(-\frac{A_1D_2 - A_2D_1}{A_1D_1 + A_2D_2}\right), \quad \ldots\ldots\ldots (28)$$

$$\delta_3 = \text{arc tan}\left(-\frac{B_1D_2 - B_2D_1}{B_1D_1 + B_2D_2}\right), \quad \ldots\ldots\ldots (29)$$

where

$$A_1 = -\bar{m}\eta^4 (1 - 3\,\zeta_1\zeta_2 + \zeta_2{}^2)/3 + \pi\eta^2 (\zeta_1 + \zeta_2)\,\text{Re}\,(\bar{a}_{12}) - \pi\eta^2\,\text{Re}\,(\bar{a}_{22}) \quad \ldots\ldots (30)$$

$$A_2 = \pi\eta^2 (\zeta_1 + \zeta_2)\,\text{Im}\,(\bar{a}_{12}) - \pi\eta^2\,\text{Im}\,(\bar{a}_{22}) \quad \ldots\ldots (31)$$

$$B_1 = -\bar{m}\eta^4\zeta_2 - \pi\eta^2 (\zeta_1 + \zeta_2)\,\text{Re}\,(\bar{a}_{11}) + \pi\eta^2\text{Re}\,(\bar{a}_{12}). \quad (32)$$

$$B_2 = -\pi\eta^2 (\zeta_1 + \zeta_2)\,\text{Im}\,(\bar{a}_{11}) + \pi\eta^2\,\text{Im}\,(\bar{a}_{12}) \quad \ldots\ldots (33)$$

$$D_1 = \bar{m}^2\eta^4 (1 + \zeta_1{}^2)/3 + \pi\bar{m}\eta^2 (1 + 4\zeta_1{}^2)\,\text{Re}\,(\bar{a}_{11})/3 - 2\,\pi\bar{m}\eta^2\,\zeta_1\,\text{Re}\,(\bar{a}_{12}) + \pi\bar{m}\eta^2\,\text{Re}\,(\bar{a}_{22}) + \pi^2\,[\text{Re}\,(\bar{a}_{11})\,\text{Re}\,(\bar{a}_{22}) - \text{Im}\,(\bar{a}_{11})\,\text{Im}\,(\bar{a}_{22}) - \text{Re}\,(\bar{a}_{12})^2 + \text{Im}\,(\bar{a}_{12})^2] \quad \ldots\ldots (34)$$

$$D_2 = \pi\bar{m}\eta^2 (1 + 4\,\zeta_1{}^2)\,\text{Im}\,(\bar{a}_{11})/3 - 2\,\pi\bar{m}\eta^2\,\zeta_1\,\text{Im}\,(\bar{a}_{12}) + \pi\bar{m}\eta^2\,\text{Im}\,(\bar{a}_{22}) + \pi^2\,[\text{Re}\,(\bar{a}_{11})\,\text{Im}\,(\bar{a}_{22}) + \text{Im}\,(\bar{a}_{11})\,\text{Re}\,(\bar{a}_{22}) - 2\,\text{Re}\,(\bar{a}_{12})\,\text{Im}\,(\bar{a}_{12})] \quad \ldots\ldots (35)$$

The non-dimensional amplitudes u and Ψ are plotted against arguments of η for various values of the parameters $\bar{m}$, ζ_1 and ζ_2 for values of υ of 0, 1/4, 1/3 and 1/2 in Figs. 4 to 21 (at the end of the paper). In general, each curve in Figs. 4 to 21 has two resonant peaks, but the first resonant peak is not clearly defined for some values of the parameters.

PERMISSIBLE AMPLITUDES OF VIBRATION

An important consideration in the design of a machine foundation is the minimizing of the detrimental amplitude of vibration, since a large amplitude may be harmful to the machine itself or to neighboring structures. Furthermore, while the vibration may not result in any mechanical or structural damage, it may cause intolerable nuisance to the people working in the vicinity of the machine. Hence, the maximum allowable amplitude must take these points into account. BARKAN (1962) found that, for normal machine operation, the amplitude computed on the basis of the permissible stresses for the supporting soil is too large to be acceptable to the operator. He presented a table of permissible amplitudes of vibration for turbogenerator

foundations. RAUSCH (1943) suggested that, to avoid damage to machines or machine foundations, the maximum velocity of the vibration should not exceed 1 in./sec, or the maximum acceleration should not exceed 0.5 times gravitational acceleration. REIHER & MEISTER (1911) observed that vibrations begin to be 'troublesome' and 'noticeable' to persons when the maximum velocity exceeds 0.1 in./sec and 0.01 in./sec respectively; these velocity criteria correspond to amplitudes of motion of 0.001 in. and 0.0001 in. respectively for a frequency of 1000 cycles/min.

Design Example

As an example of the use of Figs. 3 to 21 for the design of a machine foundation to satisfy the aforementioned criteria, the following data are chosen.

> The machine speed is 1000 rev/min and its total weight $(m+m_o)$, including that of the foundation, is 4800 lb/ft. The unbalanced weight and the design eccentricity, ε, recommended by BARKAN (1962) are respectively 300 lb/ft and 0.02 cm (or 0.00788 in.). The foundation width, $2b$, is 10 ft, and ζ_1 and ζ_2 as defined by Eq. (23) are taken as being equal to 0.5 and 0.8 respectively. For the elastic medium $c = 700$ ft/sec, $\upsilon = 1/3$ and the unit weight is 100 lb/ft.^{3}

By Eqs. (21) and (22) $\bar{m} = 1.8$ and $\eta = 0.748$. From Figs. 3, 10 and 19 the corresponding values of ν, u and Ψ obtained for $\bar{m} = 0$, 5 and 10 are respectively $\nu = 0.175$, 0.182 and 0.115; $u = 0.280$, 0.340 and 0.215; $\Psi = 0.385$, 0.325 and 0.260. By parabolic interpolation the values of the nondimensional amplitudes corresponding to $\bar{m} = 1.8$ are $\nu = 0.186$, $u = 0.323$ and $\Psi = 0.364$.

The maximum vertical displacement, ν_{max}, taken as the superposition of the amplitudes for the vertical and rocking vibrations, occurs at the tip of the foundation, and is found from Eq. (20) to be equal to 0.000519 in., while the horizontal amplitude is 0.000305 in. It can be seen that these amplitudes are smaller than the permissible values mentioned earlier. The vibration may be noticeable, however, to persons working near the machine.

DISCUSSION AND CONCLUSIONS

Reference to Eqs. (17) to (19) shows that the vertical vibration is a single-degree-of-freedom motion, while the coupled horizontal and rocking vibration is a two-degrees-of-freedom motion. Hence, each response curve has one resonance peak in Fig. 3, while two resonance peaks are shown in Figs. 4 to 21. The second peak in each of Figs. 4 to 21, beyond which the amplitude decreases rapidly, is higher in general than the first peak except for some value of υ, $\overline{m}$, ζ_1 and ζ_2. The slight increase in each response curve in Figs. 4 to 12 well beyond the second resonance may be attributed to the approximate nature of the stiffnesses used in this study.

The response of the system depends heavily on η and $\overline{m}$. The influences of ζ_1 and ζ_2 on the coupled horizontal and rocking vibration are also significant, while the influence of the Poisson's ratio is less prominent. The smaller the mass ratio, the flatter becomes the response curve. A low mass ratio, which is common in practice, gives a resonance frequency factor greater than 1.5; which is beyond the scope of this paper. The design of foundations for higher values of η requires further study. Thus, in order that the responses of high speed machine foundations may be predicted, the contact stiffnesses for high frequency factors should be investigated.

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APPENDIX : LIST OF SYMBOLS

$\bar{a}, \bar{a}_{ij}$	= complex stiffnesses; i, j = 1, 2
b	= half width of foundation
c	= shear wave velocity
exp	= exponential function, e
F	= magnitude of excitation force
F	= complex excitation force function
G	= modulus of rigidity of elastic medium
H	= height from centroid of foundation to point of application of force
h	= half thickness of foundation
Im	= imaginary part of a complex function
i	= imaginary number
J	= mass polar moment of inertia of foundation per unit length along z-axis
m	= mass of foundation per unit length along z-axis
m_o	= unbalanced mass per unit length
$\overline{m}$	= mass ratio
P_r, Q_r, T_r	= contact vertical force, contact horizontal force and contact torque, per unit length along z-axis
P,Q,T	= complex functions of contact vertical force, contact horizontal force and contact torque, per unit length along z-axis
Re	= real part of a complex function
u, v, Ψ	= non-dimensional amplitudes for horizontal, vertical and rocking vibration

u_r, v_r, Ψ_r	= horizontal displacement, vertical displacement and rocking angle, of the center of the base of the foundation
u,v,ψ	= complex displacement functions for horizontal, vertical and rocking vibrations
U,V,$\overline{\Psi}$	= amplitudes of horizontal, vertical and rocking vibrations
x,y,z	= coordinate axes
$\delta_1,\delta_2,\delta_3$	= phase angles
ε	= eccentricity of unbalanced mass m_o
ζ_1	= h/b
ζ_2	= H/b
η	= frequency factor, $b\Omega/c$
υ	= Poisson's ratio
Ω	= angular velocity
ρ	= mass density of elastic medium

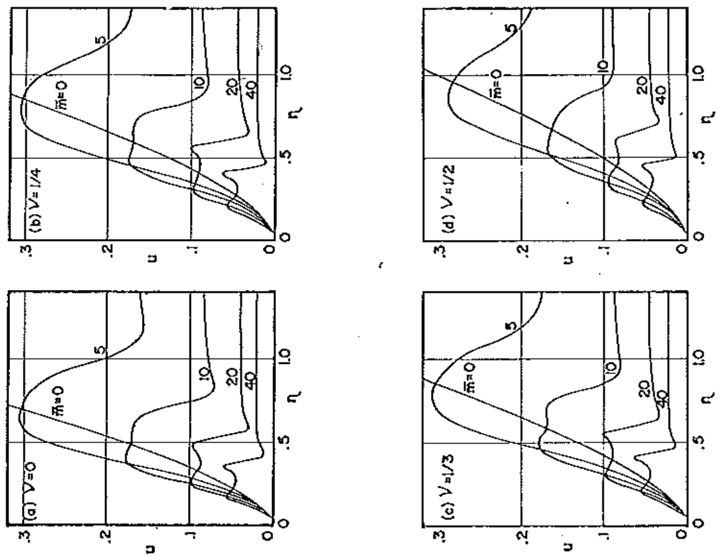

Fig. 4. Amplitudes versus frequency factors for $\zeta_1 = 0.10$ & $\zeta_2 = 0.45$ (horizontal vibration).

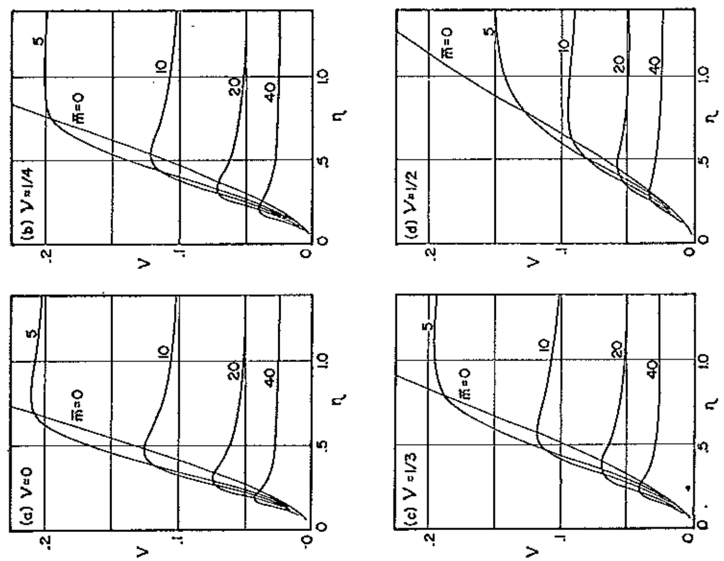

Fig. 3. Amplitudes versus frequency factors (vertical vibration).

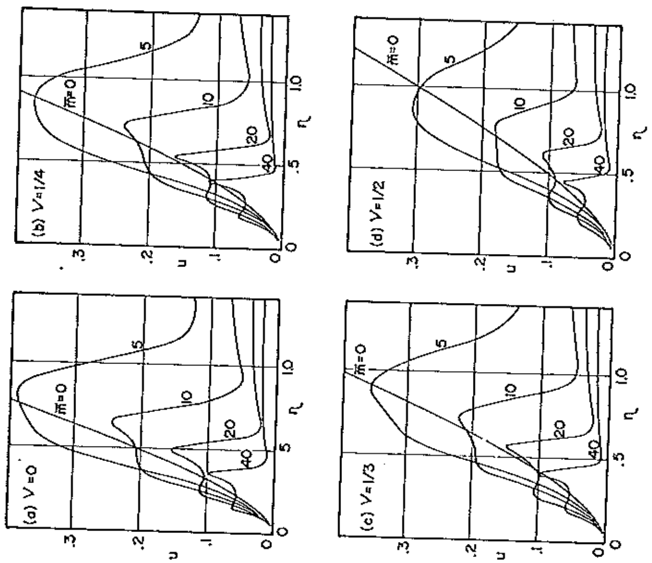

Fig. 5. Amplitudes versus frequency factors for $\zeta_1 = 0.10$ & $\zeta_2 = 0.80$ (horizontal vibration).

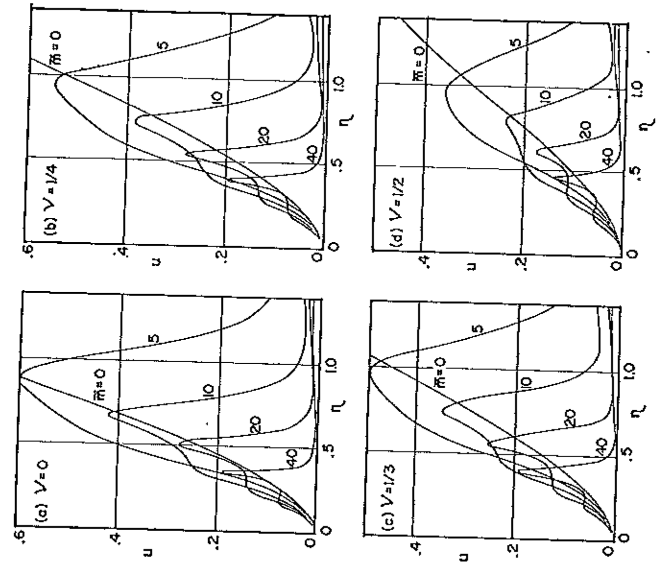

Fig. 6. Amplitudes versus frequency factors for $\zeta_1 = 0.10$ & $\zeta_2 = 1.50$ (horizontal vibration).

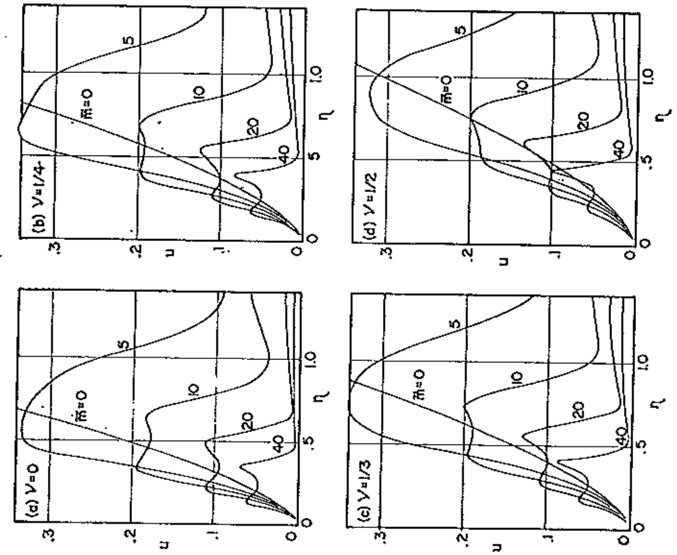

Fig. 7. Amplitudes versus frequency factors for $\zeta_1 = 0.25$ & $\zeta_2 = 0.60$ (horizontal vibration).

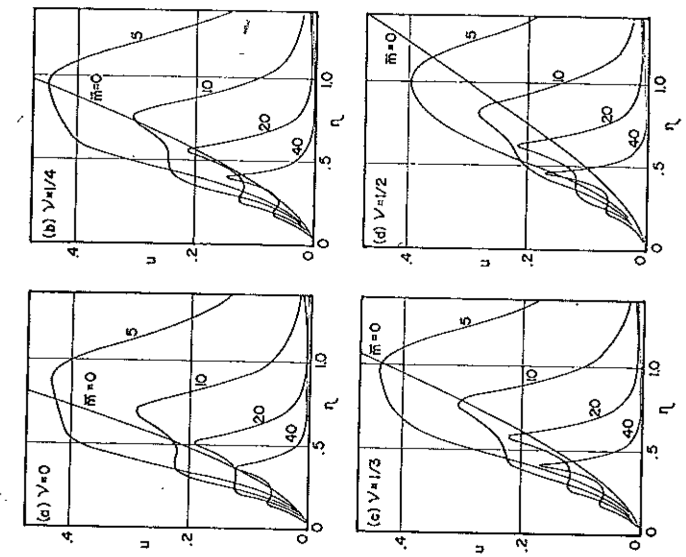

Fig. 8. Amplitudes versus frequency factors for $\zeta_1 = 0.25$ & $\zeta_2 = 1.00$ (horizontal vibration).

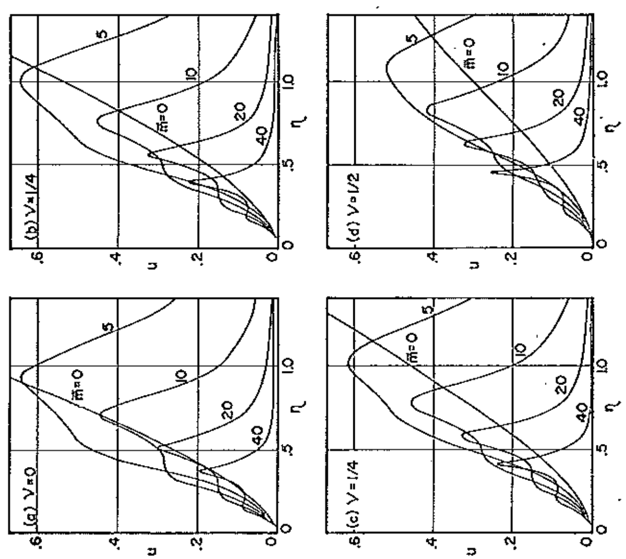

Fig. 9. Amplitudes versus frequency factors for $\zeta_1 = 0.25$ & $\zeta_2 = 1.50$ (horizontal vibration).

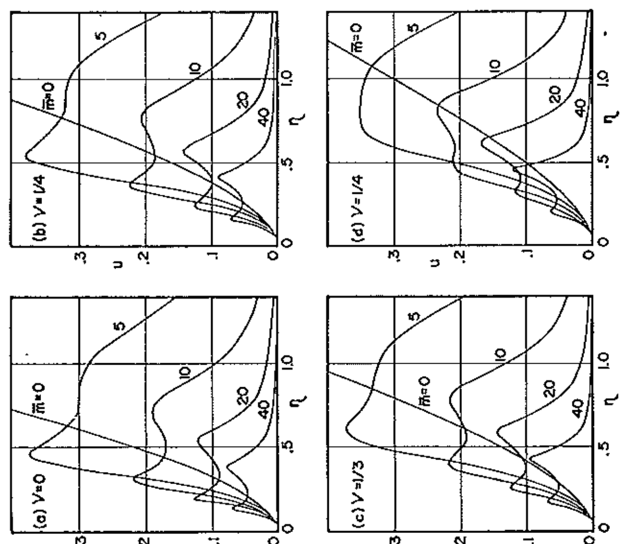

Fig. 10. Amplitudes versus frequency factors for $\zeta_1 = 0.50$ & $\zeta_2 = 0.80$ (horizontal vibration).

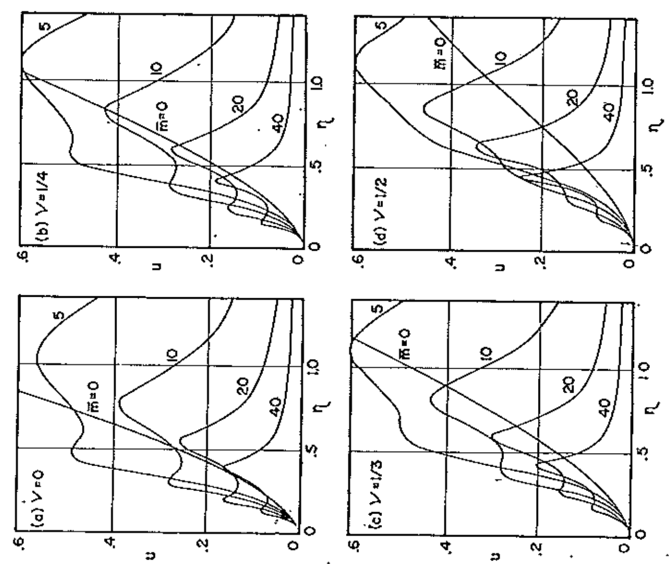

Fig. 11. Amplitudes versus frequency factors for $\zeta_1 = 0.50$ & $\zeta_2 = 1.50$ (horizontal vibration).

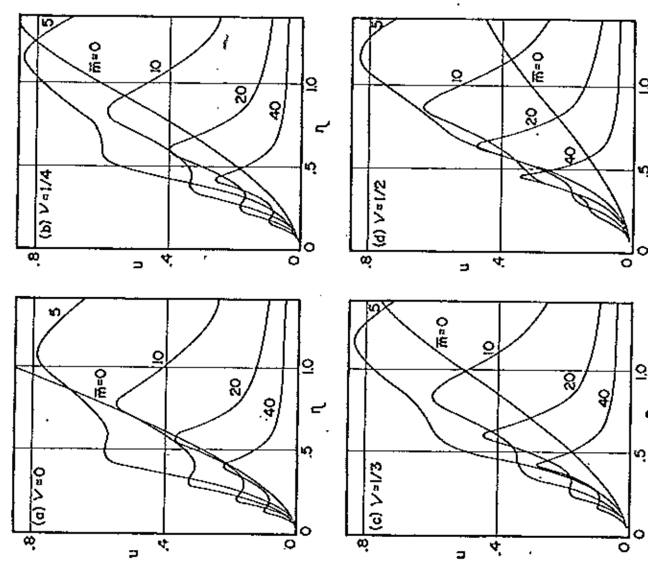

Fig. 12. Amplitudes versus frequency factors for $\zeta_1 = 0.50$ & $\zeta_2 = 2.00$ (horizontal vibration).

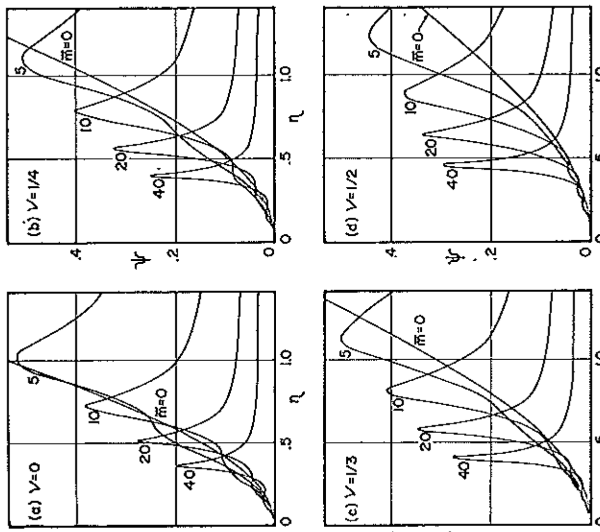

Fig. 13. Amplitudes versus frequency factors for $\zeta_1 = 0.10$ & $\zeta_2 = 0.45$ (rocking vibration).

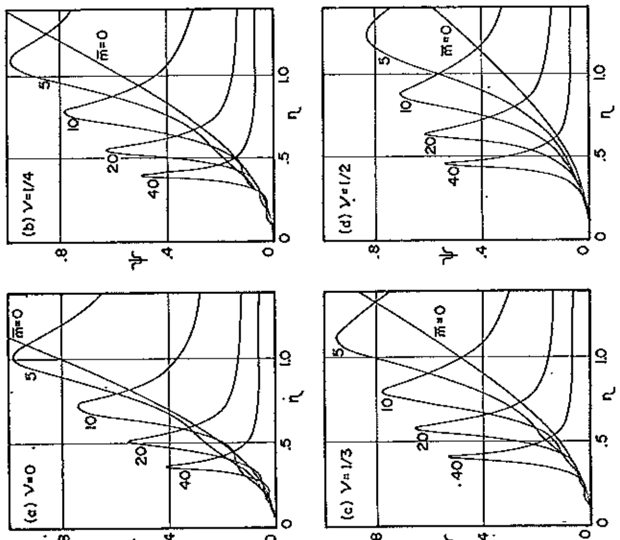

Fig. 14. Amplitudes versus frequency factors for $\zeta_1 = 0.10$ & $\zeta_2 = 0.80$ (rocking vibration).

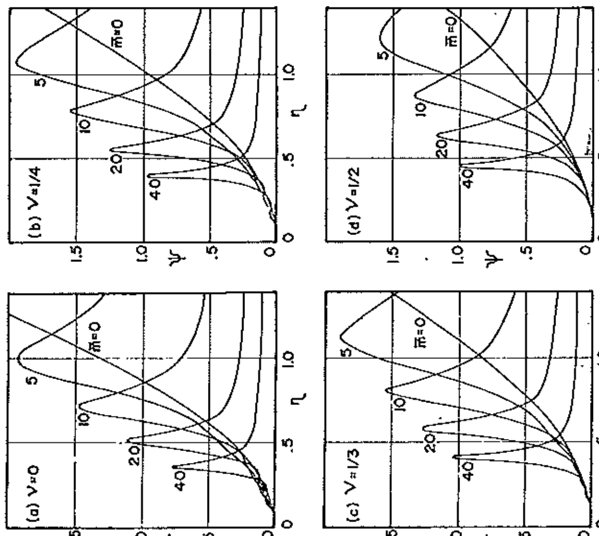

Fig. 15. Amlitudes versus frequency factors for $\zeta_1 = 0.10$ & $\zeta_2 = 1.50$ (rocking vibration).

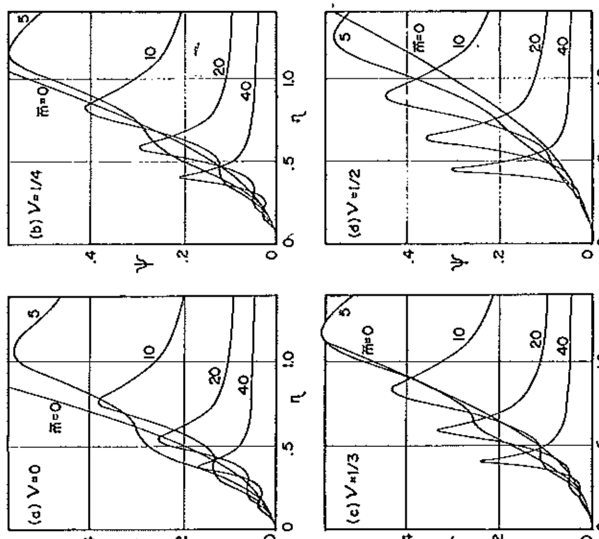

Fig. 16. Amplitudes versus frequency factors for $\zeta_1 = 0.25$ & $\zeta_2 = 0.60$ (rocking vibration).

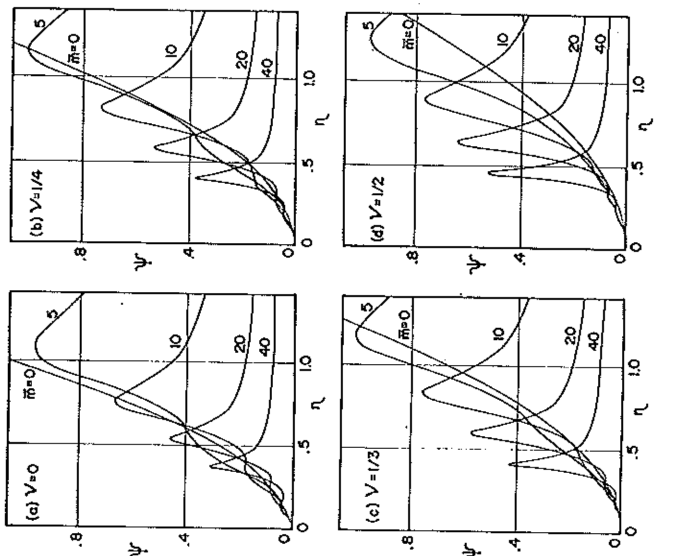

Fig. 17. Amplitudes versus frequency factors for $\zeta_1 = 0.25$ & $\zeta_2 = 1.00$ (rocking vibration).

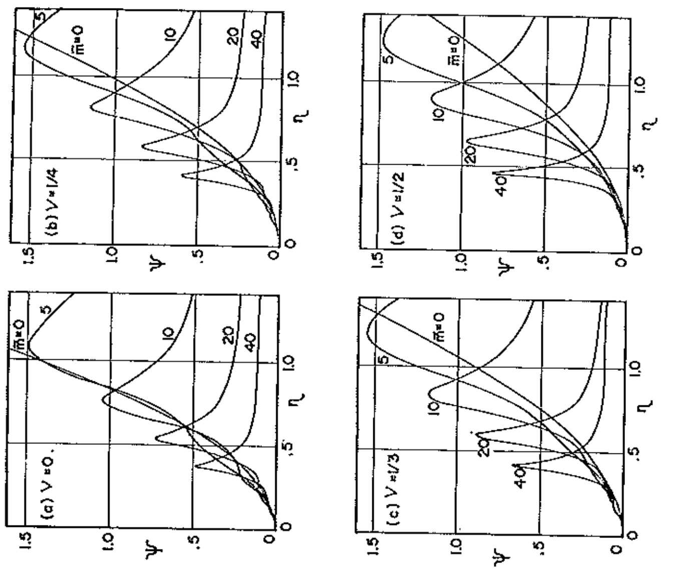

Fig. 18. Amplitudes versus frequency factors for $\zeta_1 = 0.25$ & $\zeta_2 = 1.50$ (rocking vibration).

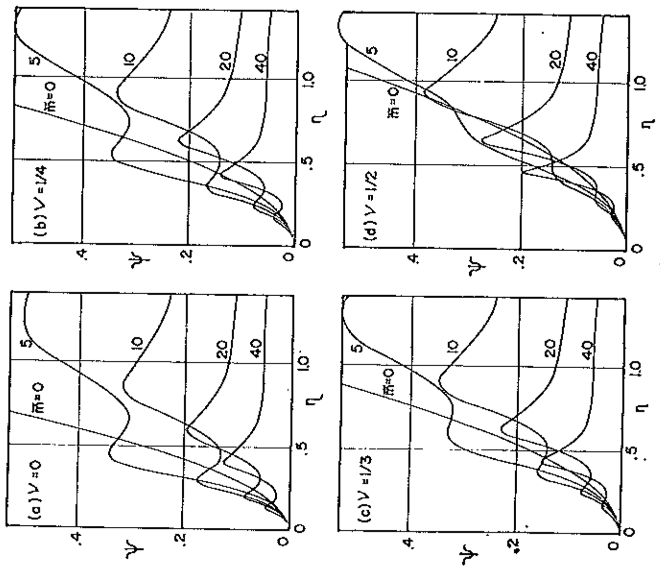

Fig. 19. Amplitudes versus frequency factors for $\zeta_1 = 0.50$ & $\zeta_2 = 0.80$ (rocking vibration).

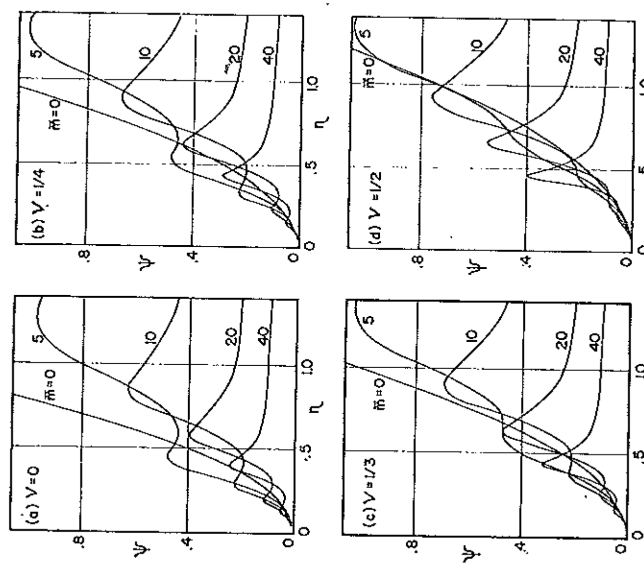

Fig. 20. Amplitudes versus frequency factors for $\zeta_1 = 0.50$ & $\zeta_2 = 1.50$ (rocking vibration).

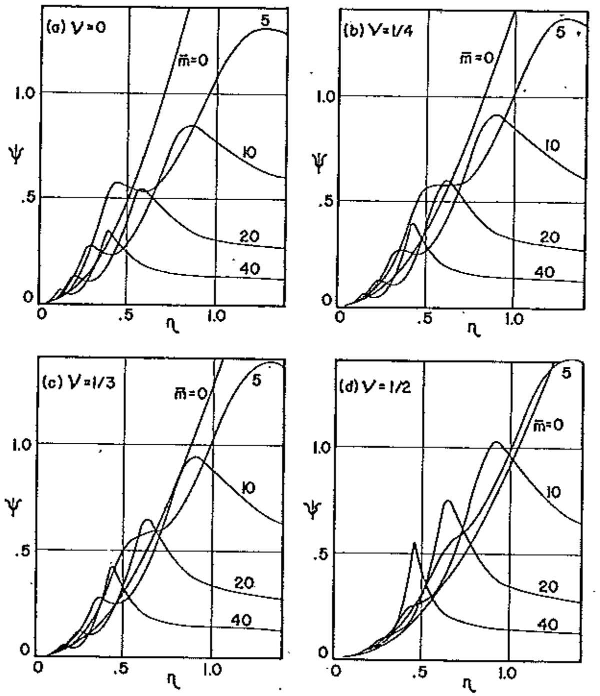

Fig. 21. Amplitudes versus frequency factors for $\zeta_1 = 0.50$ & $\zeta_2 = 2.00$ (rocking vibration).